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Vertex Models with Alternating Spins

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Dedicated to Professor Mikio Sato on his seventieth birthday.

Abstract

The diagonalisation of the transfer matrices of solvable vertex models with alternating spins is given. The crystal structure of (semi-)infinite tensor products of finite dimensional $U_q(\widehat{sl}_2)$ crystals with alternating dimensions is determined. Upon this basis the vertex models are formulated and then solved by means of $U_q(\widehat{sl}_2)$ intertwiners.

1 Introduction

In [1], the diagonalisation of the XXZ Hamiltonian,

$$H_{XXZ} \;=\; -\frac{1}{2}\sum_{k=-\infty}^{\infty}\Big(\sigma^x_{k+1}\sigma^x_k+\sigma^y_{k+1}\sigma^y_k+\Delta\sigma^z_{k+1}\sigma^z_k\Big), \tag{1.1}$$

in the anti-ferromagnetic regime ($\Delta=\frac{q+q^{-1}}{2}<-1$) was carried out by making use of the representation theory of the quantum affine algebra $U_q(\widehat{sl}_2)$. The key observation in this method was the identification of the semi-infinite tensor product of the two-dimensional representation $V^{(1)}\simeq\mathbf{C}^2$ of $U_q(\widehat{sl}_2)$ with the level one irreducible highest weight representation $V(\Lambda_i)$ $(i=0,1)$ of the same algebra [2],

$$\cdots\otimes\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathbf{C}^2\otimes\mathbf{C}^2\simeq V(\Lambda_i). \tag{1.2}$$

Using (1.2), the corner transfer matrix $A(\zeta)$ of the corresponding six-vertex model was identified with the grading operator

$$A(\zeta)\;\sim\;\zeta^{-D}, \tag{1.3}$$

and the half transfer matrix $\Phi(\zeta)$ was identified with the vertex operator

$$\Phi(\zeta):V(\Lambda_i)\to V(\Lambda_{1-i})\otimes V^{(1)}_\zeta, \tag{1.4}$$

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where $V_\zeta^{(1)}$ is the evaluation representation. The choice of $i = 0, 1$ corresponds to the choice of the boundary condition at infinity.

Under these identifications, the transfer matrix $T(\zeta)$ was identified with the composition of the vertex operators acting on the tensor product of the highest and lowest weight representations,

$$T(\zeta) \quad : \quad V(\Lambda_i) \otimes V(\Lambda_j)^* \to V(\Lambda_{1-i}) \otimes V_\zeta^{(1)} \otimes V(\Lambda_j)^* \to V(\Lambda_{1-i}) \otimes V(\Lambda_{1-j})^*, \quad (1.5)$$

and then diagonalised by making use of another vertex operator [1,3]

$$\Psi^*(\xi) \quad : \quad V_\xi^{(1)} \otimes V(\Lambda_j) \to V(\Lambda_{1-j}). \quad (1.6)$$

A similar method was also applied to other models such as the higher spin generalisation of the XXZ model [4] and the ABF models [5]. In the former, for which the local spaces are $V^{(n)} \simeq \mathbf{C}^{n+1}$, the spaces of physical states in the semi-infinite volume with the chosen boundary conditions were identified with the level n irreducible highest weight representations. On the other hand, in the latter, they were identified with the coset spaces of GKO type (see also [6]).

In this paper, we study yet another example of this sort. We consider the vertex models with alternating spins. This requires new insights, both physical and mathematical, and leads to new results in the connection between solvable lattice models and representation theory.

The origin of our study is [7], in which a spin-$\frac{1}{2}$ chain with a few higher spin components (or *impurities* in physical terms) was studied by using the vertex operator

$$\Phi^{(n-1,n)}(\zeta) : V_\zeta^{(n-1)} \otimes V(\Lambda_i) \to V(\Lambda_{1-i}) \otimes V_\zeta^{(n)}. \quad (1.7)$$

This operator explains the n-fold degeneracy of the vacuum states with a chosen boundary condition when a spin-$\frac{n}{2}$ impurity is inserted in the spin-$\frac{1}{2}$ chain. In [8], the above vertex operator was identified with the half transfer matrix of the vertex model that has semi-infinite spin-$\frac{1}{2}$ horizontal lines and a spin-$\frac{n}{2}$ vertical line. In this paper we consider a vertex model with alternating spins $\frac{m}{2}$ and $\frac{n}{2}$ $(m > n)$, and diagonalise the corresponding transfer matrices. Such models were constructed and analysed using the Bethe Ansatz in [9–12]. The first step in our solution is the identification of the semi-infinite tensor product

$$\cdots \otimes \mathbf{C}^{m+1} \otimes \mathbf{C}^{n+1} \otimes \mathbf{C}^{m+1} \otimes \mathbf{C}^{n+1} \quad (1.8)$$

having an appropriate boundary condition, with the tensor product of level $m - n$ and level n highest weight representations

$$V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}). \quad (1.9)$$

Here we set

$$\lambda_a^{(\ell)} = (\ell - a)\Lambda_0 + a\Lambda_1. \quad (1.10)$$

Formula (1.3) is again valid in this situation.

In the second step, we identify the half transfer matrices (see Figure 3) having alternating spins for the horizontal lines and spin-$\frac{n}{2}$ (Case A) or $\frac{m}{2}$ (Case B) for the vertical line, with the following vertex operators.
Case A:

$$\phi^A(\zeta) \quad : \quad V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}) \xrightarrow{\mathrm{id}\otimes\Phi^{(0,n)}(\zeta)} V(\lambda_a^{(m-n)}) \otimes V(\lambda_{n-b}^{(n)}) \otimes V_\zeta^{(n)}. \tag{1.11}$$

Case B:

$$\begin{aligned}\phi^B(\zeta) \quad : \quad & V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}) \xrightarrow{\Phi^{(0,m-n)}(\zeta)\otimes\mathrm{id}} V(\lambda_{m-n-a}^{(m-n)}) \otimes V_\zeta^{(m-n)} \otimes V(\lambda_b^{(n)}) \\ & \xrightarrow{\mathrm{id}\otimes\Phi^{(m-n,m)}(\zeta)} V(\lambda_{m-n-a}^{(m-n)}) \otimes V(\lambda_{n-b}^{(n)}) \otimes V_\zeta^{(m)},\end{aligned} \tag{1.12}$$

Finally, we have two (full) transfer matrices $T^A(\zeta)$ and $T^B(\zeta)$ for cases A and B. We can think of these operators as acting on the direct sum of the vectors spaces,

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{C}}\Big(V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}), V(\lambda_c^{(m-n)}) \otimes V(\lambda_d^{(n)})\Big) \\ &\simeq V(\lambda_c^{(m-n)}) \otimes V(\lambda_d^{(n)}) \otimes \Big(V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)})\Big)^*.\end{aligned} \tag{1.13}$$

$T^A(\zeta)$ and $T^B(\zeta)$ are mutually commuting and expressed in terms of the operators $\phi^A(\zeta)$ and $\phi^B(\zeta)$, respectively.

The operator $T^A(\zeta)$ can be viewed as the limit where the number of insertions of the higher spin components becomes infinite. Therefore, in this case, one can expect that the vacuum states are infinitely degenerate, and it is indeed so. The same is true for $T^B(\zeta)$. However, if we consider the product $T(\zeta) = T^B(\zeta)T^A(\zeta)$ the infinite degeneracy resolves, and we have a unique vacuum for a fixed boundary condition. The vacuum states are given by

$$(-q)^D \in \mathrm{End}_{\mathbf{C}}\Big(V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)})\Big). \tag{1.14}$$

The excited states are constructed upon these vacua. Consider two kinds of vertex operators with spin 0 and $\frac{1}{2}$, respectively.
Spin-0 case:

$$\psi^{(0)}(\xi) \quad : \quad V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}) \to V(\lambda_{a'}^{(m-n)}) \otimes V_\xi^{(1)} \otimes V(\lambda_b^{(n)}) \to V(\lambda_{a'}^{(m-n)}) \otimes V(\lambda_{b'}^{(n)}). \tag{1.15}$$

Spin-$\frac{1}{2}$ case:

$$\psi^{(\frac{1}{2})}(\xi) \quad : \quad V_\xi^{(1)} \otimes V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}) \to V(\lambda_{a'}^{(m-n)}) \otimes V(\lambda_b^{(n)}). \tag{1.16}$$

Acting on the vacua, the operators $\psi^{(0)}(\xi)$ and $\psi^{(\frac{1}{2})}(\xi)$ create particles with spin 0 and $\frac{1}{2}$, respectively. We give the exchange relations for these operators. The vacuum states $(-q)^D$,

the operators $\psi^{(0)}(\xi)$ and $\psi^{(\frac{1}{2})}(\xi)$ and their exchange relations are the diagonalisation data of the transfer matrix $T(\zeta)$ in the sense of the vertex operator approach [1]. From the view point of the representation theory this data gives the irreducible decomposition of the space of physical states (1.13) with respect to the action of $U_q(\widehat{sl}_2)$. We call this description of the physical space the *particle* picture in comparison with the *local* picture consisting of the alternating infinite tensor product of $\mathbf{C}^{m+1}$ and $\mathbf{C}^{n+1}$. We should say that the equivalence of the local and particle pictures is a conjecture because we have no argument for the completeness of the particle decomposition except in the crystal limit $q = 0$ (see (ii) below).

Many of the results in this paper have been announced in [13]. In this paper we give proofs for them. (On the other hand, we will not discuss the mixing of ground states, one of the main results in [13]. We have nothing to add to the result and a complete proof is already given there.) To be precise, we prove the following:

(i) A crystal isomorphism between the space of semi-infinite paths $P_{a,b}$ and the crystal $B(\lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)})$.

The crystal structure of $P_{a,b}$ represents by definition the semi-infinite tensor product of the alternating finite crystals $B^{(m)}$ and $B^{(n)}$. Therefore, the crystal isomorphism mentioned above gives supporting evidence for the conjecture that there is an isomorphism between (1.8) and (1.9). We give two proofs. The first one uses the RSOS paths which describe the highest weight vectors in $B(\lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)})$. The second proof is more direct; however, the identification of the corner transfer matrix is made only in the first proof.

(ii) Crystal decomposition of the full-infinite path spaces.

We decompose each path uniquely to a union of ground state paths patched together at the 'walls' between the ground states. Under the crystal action these walls behave like elements in the affinizations [14] of $B^{(0)}$ or $B^{(1)}$. This observation gives supporting evidence for our conjecture that the particle structure of the alternating vertex model consists of the spin-0 and spin-$\frac{1}{2}$ particles.

(iii) Commutativity of the vertex operator

$$\begin{aligned} V_\zeta^{(l)} \otimes V(\lambda_a^{(k)}) \otimes V(\Lambda_i) &\to V(\lambda_{k-a}^{(k)}) \otimes V_\zeta^{(l+k)} \otimes V(\Lambda_i) \\ &\to V(\lambda_{k-a}^{(k)}) \otimes V(\Lambda_{1-i}) \otimes V_\zeta^{(l+k+1)} \end{aligned} \tag{1.17}$$

with the DVA (deformed Virasoro algebra) actions [15] on $V(\lambda_a^{(k)}) \otimes V(\lambda_i)$ and $V(\lambda_{k-a}^{(k)}) \otimes V(\lambda_{1-i})$. This fact is used to derive the properties of the vertex operators of higher level from those of level 1.

The plan of the paper is as follows. In Section 2, the vertex models with alternating spins are formulated. The ground states and the eigenvalues of the corner transfer matrices are determined. In Section 3, the path space, i.e., the $q \to 0$ limit of the model, is studied. In Section 4, we prepare some properties of the level-1 vertex operators. In Section 5, the commutativity with the DVA is proved. The diagonalisation of the transfer matrices is

discussed in Section 6. In Section 7 we give the crystal isomorphism between the local and particle pictures. Finally, we present a brief summary of our results in Sectiion 8.

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2 The Vertex Model

In this section, we recall the definition of the alternating spin vertex model of reference [13]. We define the path space, the corner transfer matrices, and the corner transfer matrix Hamiltonian.

2.1 The R-matrices

The Boltzmann weights of our vertex model are given in terms of $U'_q(\widehat{sl}_2)$ R-matrices (as usual $U'_q(\widehat{sl}_2)$ refers to the subalgebra of $U_q(\widehat{sl}_2)$ generated by e_i, f_i, t_i $(i = 0, 1)$; our comultiplication is that of [8]). We use the spin-$\frac{n}{2}$ principal $U'_q(\widehat{sl}_2)$ evaluation module $V^{(n)}_\zeta$ defined, in terms of weight vectors $u^{(n)}_i$ $(i = 0, 1, \cdots, n)$, in Section 3.1 of [8].

In this paper, we consider the spectral parameter ζ (or $z = \zeta^2$) mainly as a generic complex number, and $U_q(\widehat{sl}_2)$ as a $\mathbf{C}$-algebra. However, in Sections 4 and 5, when we develop the theory of vertex operators, we treat the spectral parameter as an auxiliary variable. Namely, when we consider the evaluation module $V^{(n)}_\zeta$, we always extend the field of coefficients to a ring by adding ζ and ζ^{-1}. Therefore, we consider $V^{(n)}_\zeta$ as the rank $n+1$ $U'_q(\widehat{sl}_2)$ module over the extended ring.

The necessary R-matrix is given by the $U'_q(\widehat{sl}_2)$ intertwiner $R^{(\ell,\ell')}(\zeta_1/\zeta_2) : V^{(\ell)}_{\zeta_1} \otimes V^{(\ell')}_{\zeta_2} \to V^{(\ell')}_{\zeta_2} \otimes V^{(\ell)}_{\zeta_1}$ (note that $R^{(\ell,\ell')}(\zeta)$ here is $PR^{(\ell,\ell')}(\zeta)$ in the notation of [8]). We fix the normalisation by the requirement $R^{(\ell,\ell')}(\zeta) = \bar{R}^{(\ell,\ell')}(\zeta)/\kappa^{(\ell,\ell')}(\zeta)$, where $\bar{R}^{(\ell,\ell')}(\zeta)(u^{(\ell)}_0 \otimes u^{(\ell')}_0) = (u^{(\ell')}_0 \otimes u^{(\ell)}_0)$, and

$$\kappa^{(\ell,\ell')}(\zeta) = \zeta^{\min(\ell,\ell')} \frac{(q^{2+\ell+\ell'}\zeta^2\,;\,q^4)_\infty (q^{2+|\ell-\ell'|}\zeta^{-2}\,;\,q^4)_\infty}{(q^{2+\ell+\ell'}\zeta^{-2}\,;\,q^4)_\infty (q^{2+|\ell-\ell'|}\zeta^2\,;\,q^4)_\infty}.$$

This choice of normalisation has two nice consequences. The first is that the partition function per site of our vertex model is equal to 1. The second is that the R-matrix has the

properties of crossing symmetry and unitarity:

$$
\begin{aligned}
R^{(\ell,\ell')}(\zeta)^{i,j}_{i',j'} &= R^{(\ell',\ell)}(-q^{-1}\zeta^{-1})^{l'-j',i}_{l'-j,i'}, && (2.1)\\
\sum_{i',j'} R^{(\ell,\ell')}(\zeta)^{i',j'}_{i_1,j_1} R^{(\ell',\ell)}(\zeta^{-1})^{j_2,i_2}_{j',i'} &= \delta_{i_1,i_2}\delta_{j_1,j_2}. && (2.2)
\end{aligned}
$$

Here we use the components defined by

$$
R^{(\ell,\ell')}(\zeta)(u^{(\ell)}_i \otimes u^{(\ell')}_j) = \sum_{i',j'} R^{(\ell,\ell')}(\zeta)^{i,j}_{i',j'}(u^{(\ell')}_{j'} \otimes u^{(\ell)}_{i'}).
$$

We wish to give an expansion of $\bar{R}^{(\ell,\ell')}(\zeta)$ in terms of certain projectors. In order to do this it is useful to introduce a homogeneous evaluation module $(V_n)_z$ with weight vectors $v^{(n)}_i$ $(i = 0, 1, \cdots, n)$. The action of $U'_q(\widehat{sl}_2)$ on $(V_n)_z$ is given by

$$
\begin{aligned}
&f_1 v^{(n)}_j = [n-j] v^{(n)}_{j+1}, \quad e_1 v^{(n)}_j = [j] v^{(n)}_{j-1}, \quad t_1 v^{(n)}_j = q^{n-2j} v^{(n)}_j, && (2.3)\\
&f_0 = z^{-1} e_1, \quad e_0 = z f_1, \quad t_0 = t_1^{-1}. && (2.4)
\end{aligned}
$$

We shall refer to the associated $U_1 = \langle e_1, f_1, t_1 \rangle$ module as V_n. The $U'_q(\widehat{sl}_2)$-modules $(V_n)_z$ and $V^{(n)}_\zeta$ are isomorphic. The isomorphism is given by

$$
\begin{aligned}
C_n(\zeta) : V^{(n)}_\zeta &\xrightarrow{\sim} (V_n)_z, && (2.5)\\
u^{(n)}_j &\longmapsto c^{(n)}_j \zeta^j v^{(n)}_j, && (2.6)
\end{aligned}
$$

where $c^{(n)}_j = \begin{bmatrix} n \\ j \end{bmatrix}_q^{\frac{1}{2}} q^{\frac{j}{2}(n-j)}$ and we identify $\zeta^2 = z$ (in this paper, we shall use the notation $[a]_q = (q^a - q^{-a})/(q - q^{-1})$, and $[a]_q!$ and $\begin{bmatrix} a \\ b \end{bmatrix}_q$ for the standard q-factorial and q-binomial coefficients). Consider the $U'_q(\widehat{sl}_2)$ intertwiner ${}_h\bar{R}^{(\ell,\ell')}(z_1/z_2) : (V_\ell)_{z_1} \otimes (V_{\ell'})_{z_2} \longrightarrow (V_{\ell'})_{z_2} \otimes (V_\ell)_{z_1}$ defined uniquely by

$$
v^{(\ell)}_0 \otimes v^{(\ell')}_0 \longmapsto v^{(\ell')}_0 \otimes v^{(\ell)}_0. \tag{2.7}
$$

The R-matrix $\bar{R}^{(\ell,\ell')}(\zeta_1/\zeta_2)$ is given in terms of this intertwiner by

$$
\bar{R}^{(\ell,\ell')}(\zeta_1/\zeta_2) = \left(C_{\ell'}(\zeta_2)^{-1} \otimes C_\ell(\zeta_1)^{-1}\right) {}_h\bar{R}^{(\ell,\ell')}((\zeta_1/\zeta_2)^2) \left(C_\ell(\zeta_1) \otimes C_{\ell'}(\zeta_2)\right). \tag{2.8}
$$

To proceed, we note that there is a U_1 highest weight vector $\Omega_p \in V_\ell \otimes V_{\ell'}$:

$$
\Omega_p = \sum_{i=0}^{p} \frac{(-1)^i q^{(\ell+1-i)i}}{[i]_q![p-i]_q!} v^{(\ell)}_i \otimes v^{(\ell')}_{p-i}, \quad 0 \le p \le \min(\ell, \ell'), \tag{2.9}
$$

that has the properties,

$$
\begin{aligned}
&e_1 \Omega_p = 0, \quad t_1 \Omega_p = q^{\ell+\ell'-2p} \Omega_p, && (2.10)\\
&(1 \otimes e_1)\Omega_p = \Omega_{p-1}, \quad (e_1 \otimes t_1)\Omega_p = -q^{\ell'+\ell-2(p-1)} \Omega_{p-1}. && (2.11)
\end{aligned}
$$

Let $P_p^{(\ell,\ell')}$ be the unique U_1 linear map $P_p^{(\ell,\ell')}: V_\ell \otimes V_{\ell'} \longrightarrow V_{\ell'} \otimes V_\ell$ with the properties

$$P_p^{(\ell,\ell')} : \Omega_p \quad \longmapsto \quad \Omega'_p, \tag{2.12}$$

$$P_r^{(\ell,\ell')} : \Omega_p \quad \longmapsto \quad 0, \quad r \neq p, \tag{2.13}$$

where Ω'_p is the corresponding highest weight vector in $V_{\ell'} \otimes V_\ell$. Then one can follow the argument of [16] to expand $R^{(\ell,\ell')}(\zeta)$ in terms of the projectors $P_p^{(\ell,\ell')}$. We find

$$_h\bar{R}^{(\ell,\ell')}(z_1/z_2) = \sum_{p=0}^{\min\{\ell,\ell'\}} \left(\prod_{j=0}^{p-1} \frac{z - q^{\ell'+\ell-2j}}{1 - zq^{\ell'+\ell-2j}} \right) P_p^{(\ell,\ell')}, \tag{2.14}$$

where $z = z_1/z_2$.

In the definition of our vertex model, we will use the R-matrix $R^{(\ell,\ell')}(\zeta)$, with ζ and q restricted to lie in the regions $-1 < q < 0$, $1 < \zeta < -q^{-1}$. If we expand

$$\bar{R}^{(\ell,\ell')}(\zeta) = \bar{R}_0^{(\ell,\ell')} + (\zeta - 1)\bar{R}_1^{(\ell,\ell')} + O((\zeta-1)^2), \tag{2.15}$$

we find

$$\lim_{q\to 0} \bar{R}_0^{(\ell,\ell')}(u_i^{(\ell)} \otimes u_j^{(\ell')}) = \begin{cases} u_i^{(\ell')} \otimes u_j^{(\ell)} & \text{if} \quad i+j \leq \ell, \ell', \\ u_{2i+j-\ell}^{(\ell')} \otimes u_{\ell-i}^{(\ell)} & \text{if} \quad \ell \leq i+j \leq \ell', \\ u_{\ell'-j}^{(\ell')} \otimes u_{i+2j-\ell'}^{(\ell)} & \text{if} \quad \ell' \leq i+j \leq \ell, \\ u_{\ell'-\ell+i}^{(\ell')} \otimes u_{j-\ell'+\ell}^{(\ell)} & \text{if} \quad \ell, \ell' \leq i+j. \end{cases} \tag{2.16}$$

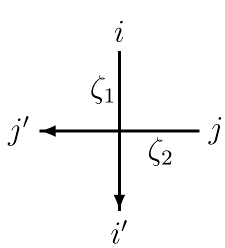
and

$$\lim_{q\to 0} \bar{R}_1^{(\ell,\ell')}(u_i^{(\ell)} \otimes u_j^{(\ell')}) = \begin{cases} (i+j)\, u_i^{(\ell')} \otimes u_j^{(\ell)} & \text{if} \quad i+j \leq \ell, \ell' \\ \ell\, u_{2i+j-\ell}^{(\ell')} \otimes u_{\ell-i}^{(\ell)} & \text{if} \quad \ell \leq i+j \leq \ell'. \\ \ell'\, u_{\ell'-j}^{(\ell')} \otimes u_{i+2j-\ell'}^{(\ell)} & \text{if} \quad \ell' \leq i+j \leq \ell. \\ (\ell'+\ell-i-j)\, u_{\ell'-\ell+i}^{(\ell')} \otimes u_{j-\ell'+\ell}^{(\ell)} & \text{if} \quad \ell, \ell' \leq i+j. \end{cases} \tag{2.17}$$

These formulas come from equations (2.8), (2.14) and the explicit formula for the projectors $P_p^{(\ell,\ell')}$ in the $q \to 0$ limit ($P_p^{(\ell,\ell')}$ become diagonal in the basis $v_i^{(\ell)} \otimes v_j^{(\ell')}$ in this limit).

The matrix element $R^{(\ell,\ell')}(\zeta_1/\zeta_2)_{i',j'}^{i,j}$ is the Boltzmann weight associated with the following configuration of spin variables $i, i' \in \{0, \cdots, \ell\}$ and $j, j' \in \{0, \cdots, \ell'\}$, and spectral parameters ζ_1 and ζ_2 around a vertex.

Figure 1

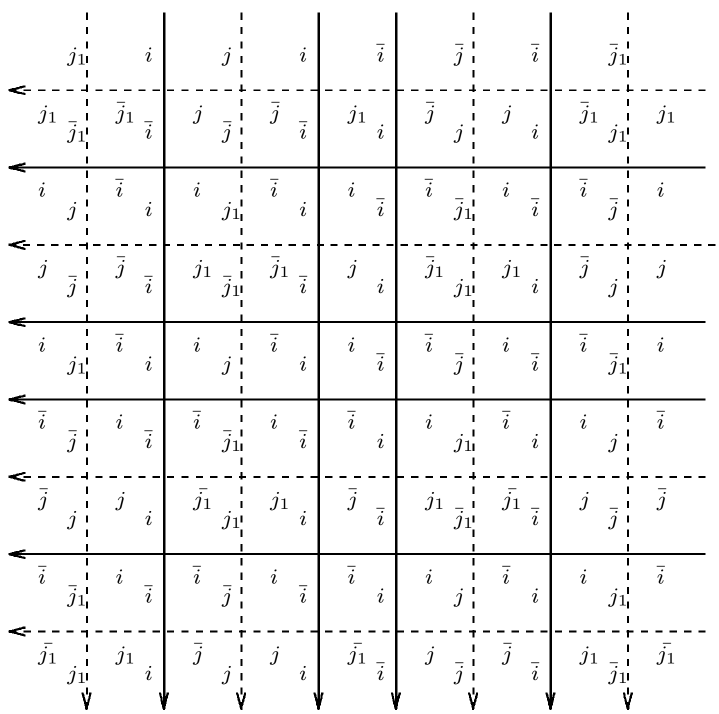


From (2.17), we see that if we choose ζ and q close to 1 and 0_- respectively, and consider the case when $\ell \leq \ell'$, then the largest Boltzmann weights will be $R^{(\ell,\ell')}(\zeta_1/\zeta_2)_{\ell-i,2i+j-\ell}^{i,j}$ with $\ell \leq i+j \leq \ell'$. Similarly for $\ell' \leq \ell$, the largest Boltzmann weights will be $R^{(\ell,\ell')}(\zeta_1/\zeta_2)_{i+2j-\ell',\ell'-j}^{i,j}$ with $\ell' \leq i+j \leq \ell$.

2.2 Definition of the vertex model

In reference [13], we define the alternating spin vertex model as the vertex model associated with the two-dimensional lattice consisting of alternating spin-$\frac{n}{2}$ and spin-$\frac{m}{2}$ lines (in both the horizontal and vertical directions), where $0 < n < m$. In fact, we choose two vertical and two horizontal spin-$\frac{n}{2}$ lines next to each other at the centre of our lattice (see Figure 2, in which the spin-$\frac{n}{2}$ and spin-$\frac{m}{2}$ lines are shown as solid and dashed lines respectively). This simplifies our discussion of the corner transfer matrix.

Figure 2



Vertical lines will carry a spectral parameter equal to ζ, and horizontal lines a spectral parameter equal to 1. We restrict our discussion to the anti-ferromagnetic region $-1 < q < 0$, $1 < \zeta < -q^{-1}$. The different local Boltzmann weights associated with the intersection vertices of this lattice are given by the R-matrices $R^{(n,m)}(\zeta)$, $R^{(m,n)}(\zeta)$, $R^{(n,n)}(\zeta)$, and $R^{(m,m)}(\zeta)$.

A ground state of such a vertex model is a configuration of the spin variables for which all of the local vertex configurations are associated with one of the largest Boltzmann weights discussed above. There are $(m-n+1)(n+1)$ different anti-ferromagnetic ground states for our model, each labelled by a pair of integers (a, b), where $0 \le a \le m-n$ and $0 \le b \le n$. The spin configuration in the (a, b) ground state is given in Figure 2 in which we use the notation $i = n-b$, $j = m-n-a+b$, $j_1 = a+b$, $\bar{i} = b$, $\bar{j} = n+a-b$, $\bar{j}_1 = m-a-b$. Define $Z_{a,b;N}$ to be the partition function (i.e., the weighted configuration sum) of such a lattice which consists of N vertices, and whose boundary spins are fixed to the values of the (a, b) ground state. With our normalisation of the R-matrices, the partition function per

unit site, $\lim_{N\to\infty} Z_{a,b;N}^{1/N}$, is equal to 1. We are interested in the infinite-volume lattice with partition function $Z_{a,b} = \lim_{N\to\infty} Z_{a,b;N}$. $Z_{a,b}$ is divergent. However, this divergence cancels for correlation functions since they are given as ratios (see [13]).

We can identify $Z_{a,b}$ with the trace of corner transfer matrices

$$Z_{a,b} = \mathrm{Tr}_{\mathcal{H}_{a,b}}(A_{NE}(\zeta)A_{SE}(\zeta)A_{SW}(\zeta)A_{NW}(\zeta)). \tag{2.18}$$

Let us explain the various elements in this formula. $\mathcal{H}_{a,b}$ is the space of eigenstates of the corner transfer matrix $A_{NW}(\zeta)$ associated with the North-West quadrant of the lattice. In the limit $q \to 0$, we can identify $\mathcal{H}_{a,b}$ with the path space $P_{a,b}$. The latter is defined to be the set of paths $|p\rangle = \cdots p(3)\,p(2)\,p(1)$ with the following restrictions:

$$p(k) \in \{0,1,\cdots,n\} \quad \text{if } k \text{ is odd}, \tag{2.19}$$

$$p(k) \in \{0,1,\cdots,m\} \quad \text{if } k \text{ is even}, \tag{2.20}$$

$$p(k) = \bar{p}(k;a,b), \quad k \gg 0, \quad \text{where} \tag{2.21}$$

$$\bar{p}(k;a,b) = \begin{cases} n-b & \text{if } k \text{ is odd;} \\ a+b & \text{if } k \equiv 0 \pmod 4; \\ m-n-a+b & \text{if } k \equiv 2 \pmod 4. \end{cases} \tag{2.22}$$

A path $|p\rangle \in P_{a,b}$ corresponds to a particular choice of the spin variables on the half-infinite column of horizontal edges running North from the centre of our lattice. The boundary condition $p(k) = \bar{p}(k;a,b)$, $k \gg 0$ corresponds to the choice of the (a,b) ground state. If $A_{NW}(\zeta)$ acts on some $|p\rangle \in P_{a,b}$, then it will produce an infinite linear combination of paths $|p'\rangle \in P_{a,b}$. One term will be of order q^0 (see (2.16)), and all the others of higher order in q. The infinite linear combination is not in $P_{a,b}$. For $q \neq 0$, $A_{NW}(\zeta)$ should be renormalised as a map $\mathcal{H}_{a,b} \to \mathcal{H}_{a,b}$, where the space $\mathcal{H}_{a,b}$ will be identified in terms of the representation theory of $U_q(\widehat{sl}_2)$ in Section 6.

The corner transfer matrices corresponding to the other quadrants can be identified as the maps $A_{SW}(\zeta) : \mathcal{H}_{a,b} \to \mathcal{H}_{m-n-a,n-b}$, $A_{SE}(\zeta) : \mathcal{H}_{m-n-a,n-b} \to \mathcal{H}_{m-n-a,n-b}$, and $A_{NE}(\zeta) : \mathcal{H}_{m-n-a,n-b} \to \mathcal{H}_{a,b}$. One can construct heuristic arguments along the lines of those in [17, 18] (which rely upon the crossing and unitarity properties of our R-matrix; given by (2.1) and (2.2) respectively), to yield the following relations among the different corner transfer matrices:

$$A_{SW}(\zeta) = CA_{NW}(-q^{-1}\zeta^{-1}),\ A_{SE}(\zeta) = CA_{NW}(\zeta)C,\ A_{NE}(\zeta) = A_{NW}(-q^{-1}\zeta^{-1})C. \tag{2.23}$$

Here, C is the 'conjugation operator': In the limit $q \to 0$, it is the operator $P_{a,b} \to P_{m-n-a,n-b}$ defined by

$$p(k) \to \begin{cases} n - p(k) & \text{if } k \text{ is odd;} \\ m - p(k) & \text{if } k \text{ is even.} \end{cases}$$

When $q \neq 0$, it will be the operator $\mathcal{H}_{a,b} \to \mathcal{H}_{m-n-a,n-b}$ which exchanges the fundamental weights $\Lambda_0 \leftrightarrow \Lambda_1$ of $U_q(\widehat{sl}_2)$.

The corner transfer matrix $A_{NW}(\zeta)$ has a remarkably simple form in the infinite-volume limit. Baxter's argument (see [19]) applied here implies $A_{NW}(\zeta) = c(\zeta)\zeta^{-H_{CTM}}$. Here, $c(\zeta)$ is a divergent scalar. H_{CTM} is the corner transfer matrix Hamiltonian, which is independent of ζ and has a non-negative integer spectrum. Using (2.23), we then have that up to a divergent scalar the infinite-volume partition function $Z_{a,b}$ is proportional to $\mathrm{Tr}_{\mathcal{H}_{a,b}}((-q)^{2H_{CTM}})$.

2.3 The corner transfer matrix Hamiltonian H_{CTM}

The corner transfer matrix Hamiltonian H_{CTM} is defined by $H_{CTM} = -\frac{d}{d\zeta}A_{NW}(\zeta)|_{\zeta=1}$: $\mathcal{H}_{a,b} \to \mathcal{H}_{a,b}$. Its action on a path $|p\rangle \in P_{a,b}$ can be calculated from (2.15):

$$H_{CTM} = -\sum_{s=1}^{\infty} s\left(H_{1;2s+1,2s,2s-1} + H_{2;2s+2,2s+1,2s} + 2H_{3;2s+1,2s}\right). \tag{2.24}$$

Here, $H_{1;2s+1,2s,2s-1}$ acts as the identity on $|p\rangle \in P_{a,b}$ except at the positions $2s+1$, $2s$, $2s-1$, where its action, written in terms of $\bar{R}_0^{(\ell,\ell')}$ and $\bar{R}_1^{(\ell,\ell')}$ as defined in (2.15), is given by

$$H_1 \;=\; (\bar{R}_0^{(m,n)} \otimes 1)(1 \otimes \bar{R}_1^{(n,n)})(\bar{R}_0^{(n,m)} \otimes 1). \tag{2.25}$$

Similarly, $H_{2;2s+2,2s+1,2s}$ acts as the identity except at the positions $2s+2$, $2s+1$, $2s$, where it acts as

$$H_2 \;=\; (1 \otimes \bar{R}_0^{(m,n)})(\bar{R}_1^{(m,m)} \otimes 1)(1 \otimes \bar{R}_0^{(n,m)}). \tag{2.26}$$

Finally, $H_{3;2s+1,2s}$ acts as the identity except at the positions $2s+1$, $2s$, where it acts as

$$H_{2s+1,2s} \;=\; \bar{R}_0^{(m,n)}\bar{R}_1^{(n,m)} = \bar{R}_1^{(m,n)}\bar{R}_0^{(n,m)}. \tag{2.27}$$

The equality of the last two expressions follows from the unitarity property (2.2).

In the limit $q \to 0$, $H_1, H_2, H_3 : P_{a,b} \to P_{a,b}$ act diagonally. Let us use the notation

$$\lim_{q\to 0} H_1(u_i^{(n)} \otimes u_j^{(m)} \otimes u_k^{(n)}) \;=\; h_1(i,j,k)(u_i^{(n)} \otimes u_j^{(m)} \otimes u_k^{(n)}), \tag{2.28}$$

$$\lim_{q\to 0} H_2(u_i^{(m)} \otimes u_j^{(n)} \otimes u_k^{(m)}) \;=\; h_2(i,j,k)(u_i^{(m)} \otimes u_j^{(n)} \otimes u_k^{(m)}), \tag{2.29}$$

$$\lim_{q\to 0} H_3(u_i^{(n)} \otimes u_j^{(m)}) \;=\; h_3(i,j)(u_i^{(n)} \otimes u_j^{(m)}). \tag{2.30}$$

Using (2.16) and (2.17), we find

$$h_1(i,j,k) = \begin{cases} \{k+j\}_n & \text{if} \quad i+j \le m,n; \\ \{k+i+2j-m\}_n & \text{if} \quad m \le i+j \le n; \\ \{k+n-i\}_n & \text{if} \quad n \le i+j \le m; \\ \{k+n-m+j\}_n & \text{if} \quad n,m \le i+j, \end{cases} \tag{2.31}$$

$$h_2(i,j,k) = \begin{cases} \{i+j\}_m & \text{if} \quad j+k \le m,n; \\ \{i+m-k\}_m & \text{if} \quad m \le j+k \le n; \\ \{i+k+2j-n\}_m & \text{if} \quad n \le j+k \le m; \\ \{i+j+m-n\}_m & \text{if} \quad n,m \le j+k, \end{cases} \tag{2.32}$$

$$h_3(i,j) = \begin{cases} i+j & \text{if} \quad i+j \le m,n; \\ m & \text{if} \quad m \le i+j \le n; \\ n & \text{if} \quad n \le i+j \le m; \\ m+n-i-j & \text{if} \quad n,m \le i+j. \end{cases} \tag{2.33}$$

Here we have used the notation

$$\{a\}_b = \begin{cases} a & \text{if} \quad a \le b; \\ 2b-a & \text{if} \quad b \le a. \end{cases} \tag{2.34}$$

In the next section we shall make use of the 'crystal energy' of a path $|p\rangle \in P_{a,b}$, which we denote by $h(p)$ and define as

$$\begin{aligned} h(p) = -\sum_{s=1}^{\infty} s \; \Big(& h_1\big(p(2s+1), p(2s), p(2s-1)\big) - h_1\big(\bar{p}(2s+1), \bar{p}(2s), \bar{p}(2s-1)\big) \\ & + h_2\big(p(2s+2), p(2s+1), p(2s)\big) - h_2\big(\bar{p}(2s+2), \bar{p}(2s+1), \bar{p}(2s)\big) \\ & + 2h_3\big(p(2s+1), p(2s)\big) - 2h_3\big(\bar{p}(2s+1), \bar{p}(2s)\big)\Big). \end{aligned} \tag{2.35}$$

Here, we have abbreviated $\bar{p}(k;a,b)$ to $\bar{p}(k)$.

3 The Path Space $P_{a,b}$

The path space $P_{a,b}$ was defined by (2.19)–(2.22) in the previous section. We shall now go on to consider this space in more detail. In particular, we wish to understand the action of $U_q(\widehat{sl}_2)$ on $P_{a,b}$ in the limit $q \to 0$. The theory which systematically describes the $q \to 0$ limit of $U_q(\widehat{sl}_2)$ was developed by Kashiwara and others, and is known as the theory of crystal bases [14,20,21]. The main content of this section is a proof of the crystal isomorphism $P_{a,b} \simeq B(\lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)})$. Here, $\lambda_j^{(k)} = (k-j)\Lambda_0 + j\Lambda_1$, $j \in \{0,1,\cdots,k\}$, is a level k dominant integral weight and $B(\lambda_j^{(k)})$ is the crystal associated with the highest weight module $V(\lambda_j^{(k)})$

(see [20]). We shall use a principal grading operator D, defined on $B(\lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)})$ by

$$D = -\rho + (\rho, \lambda_a^{(m-n)} + \lambda_b^{(n)}), \tag{3.1}$$

where $\rho = \Lambda_0 + \Lambda_1$ and $(\cdot,\cdot)$ is the symmetric bilinear form used in [3]. We denote by $B^{(k)}$ the crystal of the $k+1$ dimensional $U_q'(\widehat{sl}_2)$ module $(V_k)_z$ with $z = 1$. Set $\sigma\lambda_j^{(k)} = j\Lambda_0 + (k-j)\Lambda_1$.

We give two proofs. The first makes use of a relation between our models and the fusion RSOS models. The second proceeds by examining the crystal isomorphism

$$B(\lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)}) \simeq B(\sigma^N \lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)}) \otimes (B^{(m)} \otimes B^{(n)})^{\otimes N}, \tag{3.2}$$

where $N \in \mathbf{Z}_{>0}$.

3.1 Identification of $P_{a,b}$ with the tensor product of crystals with highest weights

Let us give the rules for the crystal action of $\tilde{f}_i, \tilde{e}_i$ $(i = 0, 1)$ on a path $|p\rangle \in P_{a,b}$ (for the definition of $\tilde{f}_i, \tilde{e}_i$ and for a detailed discussion of the theory of crystal bases, see [14,20,22]): First, for each $k > 0$, replace each $p(k)$ by the sequence of 1's and 0's

$$p(k) \to \underbrace{1\cdots1}_{\#1}\ \underbrace{0\cdots0}_{\#0}\,, \tag{3.3}$$

where

$$(\#1, \#0) = \begin{cases} (n - p(k), p(k)) & \text{for } i = 0,\ k \text{ odd,} \\ (m - p(k), p(k)) & \text{for } i = 0,\ k \text{ even,} \\ (p(k), n - p(k)) & \text{for } i = 1,\ k \text{ odd,} \\ (p(k), m - p(k)) & \text{for } i = 1,\ k \text{ even.} \end{cases} \tag{3.4}$$

Then, remove repeatedly all occurrences of adjacent 01 pairs until we have a sequence of the form $1\cdots1\,0\cdots0$. On the remaining sequence, use the rule

$$\tilde{f}_i(\underbrace{1\cdots1}_{j}\underbrace{0\cdots0}_{k}) = \underbrace{1\cdots1}_{j+1}\underbrace{0\cdots0}_{k-1}, \tag{3.5}$$

$$\tilde{e}_i(\underbrace{1\cdots1}_{j}\underbrace{0\cdots0}_{k}) = \underbrace{1\cdots1}_{j-1}\underbrace{0\cdots0}_{k+1}. \tag{3.6}$$

Finally, put the 01 pairs back into their original positions and rebuild the modified path using the inverse of the replacement given in (3.3).

If we remove 01 from the sequence $\{\bar{p}(k; a, b)\}_{k \geq k_0}$, then for $i = 1$, we get the sequences, $\underbrace{0\cdots0}_{a+b}$ (if $k_0 \equiv 1 \pmod 4$), $\underbrace{0\cdots0}_{n+a-b}$ (if $k_0 \equiv 2 \pmod 4$), $\underbrace{0\cdots0}_{m-n-a+b}$ (if $k_0 \equiv 3 \pmod 4$), $\underbrace{0\cdots0}_{m-a-b}$

(if $k_0 \equiv 4 \pmod 4$)). In all cases, $\tilde{e}_1$ annihilates the sequence. For $i=0$, the same is true with the replacement of a by $m-n-a$, and of b by $n-b$.

For $k \geq 1$, we use the notation

$$\mathrm{wt}_k(p) = \begin{cases} (n-2p)(\Lambda_1 - \Lambda_0) & \text{if } k \text{ is odd;} \\ (m-2p)(\Lambda_1 - \Lambda_0) & \text{if } k \text{ is even.} \end{cases} \tag{3.7}$$

Definition 3.1. *A path $|p\rangle \in P_{a,b}$ is called admissible if the sequence of weights $\{\lambda(k)\}_{k\geq 1}$ defined by*

$$\begin{aligned} \lambda(k+1) + \mathrm{wt}_k(p(k)) &= \lambda(k), & (3.8)\\ \lambda(1) &= \lambda^{(m)}_{a+b} + \sum_{k\geq 1}\Big(\mathrm{wt}_k(p(k)) - \mathrm{wt}_k(\bar{p}(k;a,b))\Big), & (3.9) \end{aligned}$$

satisfies

$$\langle h_1, \lambda(k+1)\rangle \geq p(k), \quad \langle h_0, \lambda(k+1)\rangle \geq \begin{cases} n - p(k) & \textit{if } s \textit{ is odd;} \\ m - p(k) & \textit{if } s \textit{ is even.} \end{cases} \tag{3.10}$$

If $|p\rangle$ is admissible, we have $\lambda(k) \in \{\lambda^{(m)}_j; 0 \leq j \leq m\}$. Note that the path $|\bar{p}\rangle \in P_{a,b}$ is admissible, and that the corresponding sequence of weights is given by the period 4 repetition

$$\cdots \lambda^{(m)}_{m-a-b}\, \lambda^{(m)}_{m-n-a+b}\, \lambda^{(m)}_{n+a-b}\, \lambda^{(m)}_{a+b}.$$

With these definitions in hand, we can proceed to state and prove the following theorem:

Theorem 3.2. *There is a crystal isomorphism $P_{a,b} \simeq B(\lambda^{(m-n)}_a) \otimes B(\lambda^{(n)}_b)$, under which the principal grading is given by $D|p\rangle = h(p)|p\rangle$.*

The proof is given after preparing the following lemma.

Lemma 3.3. *A path $|p\rangle \in P_{a,b}$ is highest, i.e., $\tilde{e}_i|p\rangle = 0$, for $i = 0, 1$, if and only if it is admissible.*

Proof. First note that the tensor product rule for crystals (see [20]) implies that if a path $|p\rangle \in P_{a,b}$ is highest, and if we split the tensor product expression for the path at any arbitrary point l to write

$$|p\rangle = \big(\cdots \otimes p(l+2) \otimes p(l+1) \otimes p(l)\big) \otimes \big(p(l-1) \otimes p(l-2) \otimes \cdots p(1)\big), \tag{3.11}$$

then $\big(\cdots \otimes p(l+2) \otimes p(l+1) \otimes p(l)\big)$ must also be highest.

Now suppose $l \gg 0$ such that $p(k) = \bar{p}(k;a,b)$ for $k \geq l$. Then, the reduction of the sequence $(p(k))_{k\geq l}$ for $i = 0,1$ gives rise to $\underbrace{0\cdots 0}_{\langle h_i, \bar{\lambda}(l;a,b)\rangle}$. Since $|p\rangle$ is highest, the path $(p(k))_{k\geq l-1}$ must also be highest. For this to be true, it is necessary and sufficient that

$$\langle h_1, \bar{\lambda}(l;a,b)\rangle \geq p(l-1) \quad \langle h_0, \bar{\lambda}(l;a,b)\rangle \geq \begin{cases} n - p(l-1) & \text{if } l \text{ is even ;} \\ m - p(l-1) & \text{if } l \text{ is odd .} \end{cases} \tag{3.12}$$

Namely, we have (3.10) for $k = l - 1$. Setting $\lambda(l-1) = \bar{\lambda}(l;a,b) + \mathrm{wt}(p(l-1))$ we can repeat this argument. Continuing in the same way to $\lambda(l-2), \lambda(l-3)$, etc., we can prove the lemma. $\square$

Proof of Theorem 3.2. First let us consider the conditions (3.10) in more detail. If we write $\lambda(k) = \lambda^{(m)}_{a(k)}$ (where $a(k) \in \{0, 1, \cdots m\}$), then the conditions for k odd become

$$
\begin{aligned}
a(k+1) + (n - 2p(k)) &= a(k), && (3.13)\\
a(k+1) &\geq p(k), && (3.14)\\
m - a(k+1) &\geq n - p(k). && (3.15)
\end{aligned}
$$

Eliminating $p(k)$, we find

$$
\begin{aligned}
a(k+1) - a(k) &\in \{-n, -n+2, \cdots, n\}, && (3.16)\\
n \leq \quad a(k+1) + a(k) &\leq 2m - n. && (3.17)
\end{aligned}
$$

On the other hand, the admissibility conditions for k even become

$$
\begin{aligned}
a(k+1) + (m - 2p(k)) &= a(k), && (3.18)\\
a(k+1) &\geq p(k), && (3.19)\\
m - a(k+1) &\geq m - p(k). && (3.20)
\end{aligned}
$$

Eliminating $p(k)$ gives just

$$
a(k+1) = m - a(k). \tag{3.21}
$$

From these considerations, it follows that an admissible sequence of weights can be written in the form

$$
\cdots \sigma(\lambda^{(m)}_{r(5)}) \quad \sigma(\lambda^{(m)}_{r(4)}) \quad \lambda^{(m)}_{r(4)} \quad \lambda^{(m)}_{r(3)} \quad \sigma(\lambda^{(m)}_{r(3)}) \quad \sigma(\lambda^{(m)}_{r(2)}) \quad \lambda^{(m)}_{r(2)} \quad \lambda^{(m)}_{r(1)}, \tag{3.22}
$$

where the path $|r\rangle = \cdots r(4)\, r(3)\, r(2)\, r(1)$ lies in the space $R_{a,b}$, defined as the set of paths for which

$$
\begin{aligned}
& r(k) \in \{0, \cdots, m\}, \\
& r(k+1) - r(k) \in \{-n, -n+2, \cdots, n\}, && (3.23)\\
& n \leq r(k+1) + r(k) \leq 2m - n, \\
& r(k) = \bar{r}(k;a,b), \quad k \gg 0, \quad \text{where} && (3.24)
\end{aligned}
$$

$$
\bar{r}(k;a,b) = \begin{cases} a+b & k \text{ odd}; \\ a+n-b & k \text{ even}. \end{cases} \tag{3.25}
$$

That is, we can identify an admissible path $|p\rangle \in P_{a,b}$ with a path $|r\rangle \in R_{a,b}$ by defining

$$r(k) = \begin{cases} a(2k-1) & \text{if } k \text{ is odd;} \\ m - a(2k-1) & \text{if } k \text{ is even.} \end{cases} \tag{3.26}$$

The restrictions on $|r\rangle \in R_{a,b}$ are those on the space of states of the $U_q(\widehat{sl}_2)$ fusion RSOS models. In [5], such a model is labelled by two integers (ℓ, N) and by the level $k = \ell + N$. The connection with our notation is that $(\ell, N, k) \leftrightarrow (m-n, n, m)$.

Let us denote by $\Omega\big(B(\lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)})\big)$ the space of highest weight elements in the crystal $B(\lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)})$. Then, it is a well-known theorem that

$$\begin{aligned} R_{a,b} &\simeq \Omega\big(B(\lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)})\big), \quad \text{where} && (3.27) \\ D|r\rangle &= \left(\frac{1}{2}\sum_{k>0} k|r(k+2) - r(k)|\right)|r\rangle. && (3.28) \end{aligned}$$

This theorem appears at least implicitly in many of the original works on RSOS models (see [23, 24] for example). A statement and proof using the language of crystals is given in [25].

We now require two lemmas concerning the crystal energy $h(p)$ of a path in $P_{a,b}$.

Lemma 3.4. *The crystal energy of an admissible path $|p\rangle \in P_{a,b}$ is given by*

$$h(p) = \frac{1}{2}\sum_{k=1}^{\infty} k|r(k+2) - r(k)|. \tag{3.29}$$

Proof. The crystal energy $h(p)$ of a path is defined by (2.31)–(2.35). If $|p\rangle$ is admissible we have

$$\begin{aligned} a(2k+2) + n - 2p(2k+1) &= a(2k+1), && (3.30) \\ a(2k+1) + m - 2p(2k) &= a(2k). && (3.31) \end{aligned}$$

Adding these equations and using (3.17) and (3.21) gives $n \le p(2k+1) + p(2k) \le m$. Hence h_1, h_2 and h_3 are given by

$$\begin{aligned} h_1(p(2k+1), p(2k), p(2k-1)) &= \{p(2k-1) + n - p(2k+1)\}_n, && (3.32) \\ h_2(p(2k+2), p(2k+1), p(2k)) &= \{p(2k+2) + p(2k) + 2p(2k+1) - n\}_m, && (3.33) \\ h_3(p(2k+1), p(2k)) &= n. && (3.34) \end{aligned}$$

Using (3.13)–(3.21) it is simple to show that

$$\begin{aligned} p(2k-1) + n - p(2k+1) &= n + a(2k+3) - a(2k-1), && (3.35) \\ p(2k+2) + p(2k) + 2p(2k+1) - n &= m. && (3.36) \end{aligned}$$

Writing $a(2k-1)$ in terms of $r(k)$ using (3.26) then gives

$$h_1(p(2k+1), p(2k), p(2k-1)) = n - \frac{1}{2}|r(k+2) - r(k)|, \tag{3.37}$$

which completes the proof. $\square$

Lemma 3.5. *The action of $\tilde{f}_i$ on a path increases $h(p)$ by 1, and that of $\tilde{e}_i$ decreases it by 1.*

Proof. $\tilde{f}_1$ acts on a path $|p\rangle \in P_{a,b}$ by changing $p(k) \to p(k)+1$ at a single value of k. Suppose this happens for $k = 2l+1$. Then the following inequalities must hold:

$$n - p(2l+1) > p(2l), \tag{3.38}$$

$$m - p(2l+2) \leq p(2l+1). \tag{3.39}$$

Using these inequalities, (2.31)–(2.33), and the property

$$\{a+1\}_n - \{a\}_n = \begin{cases} 1 & \text{if } a < n, \\ -1 & \text{if } a \geq n, \end{cases} \tag{3.40}$$

we arrive at

$$\begin{aligned} h_1(p(2l+3), p(2l+2), p(2l+1)+1) - h_1(p(2l+3), p(2l+2), p(2l+1)) &= -1, \\ h_2(p(2l+2), p(2l+1)+1, p(2l)) - h_2(p(2l+2), p(2l+1), p(2l)) &= -1, \\ h_1(p(2l+1)+1, p(2l), p(2l-1)) - h_1(p(2l+1), p(2l), p(2l-1)) &= 0, \\ h_3(p(2l+1)+1, p(2l)) - h_3(p(2l+1), p(2l)) &= 1. \end{aligned}$$

From this we see that $h(p) \to h(p)+1$ when we change $p(2l+1) \to p(2l+1)+1$. The proofs for the case when $k = 2l$, and for $\tilde{f}_0, \tilde{e}_0, \tilde{e}_1$ are similar. $\square$

We have shown that the space of highest paths in $P_{a,b}$ is isomorphic to the space $R_{a,b}$ which is in turn isomorphic to $\Omega\big(B(\lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)})\big)$. Combining this result with (3.28), and Lemmas 3.4 and 3.5 completes the proof of Theorem 3.2. $\square$

3.2 The crystal structure of the path space

In this subsection, we will give another proof of the crystal isomorphism $P_{a,b} \cong B(\lambda) \otimes B(\mu)$ between the path space $P_{a,b}$ and the tensor product of the crystals $B(\lambda) \otimes B(\mu)$, where $\lambda = a\Lambda_1 + (m-n-a)\Lambda_0$ and $\mu = b\Lambda_1 + (n-b)\Lambda_0$. We first recall some of the fundamental results on the crystals for the quantum affine algebra $U_q(\widehat{sl}_2)$ (cf. [14]).

Let $\ell > 0$ be a positive integer and let $B^{(\ell)} = \{[j]^{(\ell)} \mid 0 \leq j \leq \ell\}$ be the perfect crystal of level ℓ for the quantum affine algebra $U_q(\widehat{sl}_2)$. The crystal structure of $B^{(\ell)}$ is given in the following:

$$[0]^{(\ell)} \underset{0}{\overset{1}{\rightleftarrows}} [1]^{(\ell)} \underset{0}{\overset{1}{\rightleftarrows}} \quad \cdots\cdots \quad \underset{0}{\overset{1}{\rightleftarrows}} [\ell-1]^{(\ell)} \underset{0}{\overset{1}{\rightleftarrows}} [\ell]^{(\ell)}$$

We will write $[j]$ in place of $[j]^{(\ell)}$ whenever there is no danger of confusion.

The following theorem gives one of the most fundamental isomorphisms in the theory of crystals for the quantum affine algebra $U_q(\widehat{sl}_2)$.

Theorem 3.6 (cf. [14]). *For any dominant integral weight $\lambda = s\Lambda_1 + (\ell - s)\Lambda_0$ of level $\ell > 0$, there exists a crystal isomorphism*

$$\psi = \psi_\lambda : B(\lambda) \xrightarrow{\sim} B(\sigma\lambda) \otimes B^{(\ell)} \tag{3.41}$$

such that

$$u_\lambda \mapsto u_{\sigma\lambda} \otimes [\ell - s], \tag{3.42}$$

where u_λ is the highest weight element of $B(\lambda)$.

Let $|p_\lambda\rangle = (p_\lambda(k))_{k=1}^{\infty}$ be the sequence of elements in $B^{(\ell)}$ whose terms are given by

$$p_\lambda(k) = \begin{cases} [\ell - s] & \text{if } k \text{ is odd,} \\ [s] & \text{if } k \text{ is even.} \end{cases} \tag{3.43}$$

For each positive integer $N > 0$, there exists a crystal isomorphism

$$\psi^{(N)} = \psi_\lambda^{(N)} : B(\lambda) \xrightarrow{\sim} B(\sigma^N\lambda) \otimes (B^{(\ell)})^{\otimes N} \tag{3.44}$$

such that

$$u_\lambda \mapsto u_{\sigma^N\lambda} \otimes p_\lambda(N) \otimes \cdots \otimes p_\lambda(2) \otimes p_\lambda(1). \tag{3.45}$$

A sequence $|p\rangle = (p(k))_{k=1}^{\infty}$ with $p(k) \in B^{(\ell)}$ is called a λ-*path in* $B^{(\ell)}$ if $p(k) = p_\lambda(k)$ for all sufficiently large k. Let $P(\lambda)$ be the set of all λ-paths in $B^{(\ell)}$. Each λ-path $|p\rangle = (p(k))_{k=1}^{\infty}$ is understood as the half-infinite tensor product $|p\rangle = \cdots \otimes p(k+1) \otimes p(k) \otimes \cdots \otimes p(2) \otimes p(1)$ and hence the set $P(\lambda)$ is given a crystal structure for the quantum affine algebra $U_q(\widehat{sl}_2)$ by the tensor product rule for the crystals.

Moreover, one can prove:

Theorem 3.7 (cf. [14]). *For each $b \in B(\lambda)$, there exists a positive integer $N > 0$ such that*

$$\psi^{(N)}(b) \in u_{\sigma^N\lambda} \otimes (B^{(\ell)})^{\otimes N}. \tag{3.46}$$

Hence we have the crystal isomorphism $B(\lambda) \xrightarrow{\sim} P(\lambda)$.

In [26], in his study of 6 vertex models of inhomogeneous type, Nakayashiki considered the crystal isomorphism ψ in a more general setting.

Theorem 3.8 (cf. [26, 27]). *Let $\mu = b\Lambda_1 + (n-b)\Lambda_0$ be a dominant integral weight of level $n > 0$ and let $B^{(k)}$ be the perfect crystal of level $k > 0$ for the quantum affine algebra $U_q(\widehat{sl}_2)$. Then there exists a crystal isomorphism*

$$\Psi : B^{(k)} \otimes B(\mu) \xrightarrow{\sim} B(\sigma\mu) \otimes B^{(n+k)} \tag{3.47}$$

such that

$$[j]^{(k)} \otimes u_\mu \mapsto u_{\sigma\mu} \otimes [j+n-b]^{(n+k)}. \tag{3.48}$$

Suppose $m > n$ and let $\lambda = a\Lambda_1 + (m-n-a)\Lambda_0$ be a dominant integral weight of level $m-n$. Define a crystal isomorphism

$$\begin{aligned} \Phi = \Phi_{\lambda,\mu} = \Psi \circ (\psi_\lambda \otimes \psi_\mu) = (\mathrm{id} \otimes \Psi \otimes \mathrm{id}) \circ (\psi_\lambda \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \psi_\mu) : \\ B(\lambda) \otimes B(\mu) \xrightarrow{\mathrm{id} \otimes \psi} B(\lambda) \otimes B(\sigma\mu) \otimes B^{(n)} \\ \xrightarrow{\psi \otimes \mathrm{id} \otimes \mathrm{id}} B(\sigma\lambda) \otimes B^{(m-n)} \otimes B(\sigma\mu) \otimes B^{(n)} \\ \xrightarrow{\mathrm{id} \otimes \Psi \otimes \mathrm{id}} B(\sigma\lambda) \otimes B(\mu) \otimes B^{(m)} \otimes B^{(n)} \end{aligned} \tag{3.49}$$

to be the composite of the crystal isomorphisms defined in Theorem 3.6 and Theorem 3.8.

By repeating the crystal isomorphism Φ, we obtain the crystal isomorphism

$$\Phi^{(2)} : B(\lambda) \otimes B(\mu) \xrightarrow{\sim} B(\lambda) \otimes B(\mu) \otimes (B^{(m)} \otimes B^{(n)})^{\otimes 2}$$

such that

$$u_\lambda \otimes u_\mu \mapsto u_\lambda \otimes u_\mu \otimes [a+b]^{(m)} \otimes [n-b]^{(n)} \otimes [m-n-a+b]^{(m)} \otimes [n-b]^{(n)}.$$

In general, let $|p_{\lambda,\mu}\rangle = (p_{\lambda,\mu}(k))_{k=1}^{\infty}$ be the sequence of elements in $B^{(m)}$ and $B^{(n)}$ whose terms are given by

$$p_{\lambda,\mu}(k) = \begin{cases} [n-b] & \text{if } k \text{ is odd,} \\ [a+b] & \text{if } k \equiv 0 \pmod 4, \\ [m-n-a+b] & \text{if } k \equiv 2 \pmod 4. \end{cases} \tag{3.50}$$

For each positive integer $N > 0$, we have a crystal isomorphism

$$\Phi^{(N)} = \Phi^{(N)}_{\lambda,\mu} : B(\lambda) \otimes B(\mu) \xrightarrow{\sim} B(\sigma^N\lambda) \otimes B(\mu) \otimes (B^{(m)} \otimes B^{(n)})^{\otimes N} \tag{3.51}$$

such that

$$u_\lambda \otimes u_\mu \mapsto u_{\sigma^N\lambda} \otimes u_\mu \otimes p_{\lambda,\mu}(2N) \otimes p_{\lambda,\mu}(2N-1) \otimes \cdots \otimes p_{\lambda,\mu}(2) \otimes p_{\lambda,\mu}(1). \tag{3.52}$$

A sequence $|p\rangle = (p(k))_{k=1}^{\infty}$ of elements in $B^{(m)}$ and $B^{(n)}$ is called a (λ,μ)-*path* if $p(k) = p_{\lambda,\mu}(k)$ for all sufficiently large k. Let $P(\lambda,\mu)$ be the set of all (λ,μ)-paths. The crystal structure of $P(\lambda,\mu)$ is the same as that of the path space $P_{a,b}$.

Now, we would like to show that there exists a crystal isomorphism $B(\lambda) \otimes B(\mu) \xrightarrow{\sim} P(\lambda,\mu)$. As in the case with the crystal isomorphism $B(\lambda) \xrightarrow{\sim} P(\lambda)$, it suffices to prove that for each $v \otimes w \in B(\lambda) \otimes B(\mu)$, there exists a positive integer $N > 0$ such that

$$\Phi^{(N)}(v \otimes w) \in u_{\sigma^N \lambda} \otimes u_\mu \otimes (B^{(m)} \otimes B^{(n)})^{\otimes N}. \tag{3.53}$$

For this purpose, we need an explicit description of the crystal isomorphism $R : B^{(m)} \otimes B^{(n)} \xrightarrow{\sim} B^{(n)} \otimes B^{(m)}$, called the *combinatorial R-matrix*, normalised by the condition $R([0]^{(m)} \otimes [0]^{(n)}) = [0]^{(n)} \otimes [0]^{(m)}$. We rephrase (2.16) as follows.

Lemma 3.9. *The normalised combinatorial R-matrix $R : B^{(m)} \otimes B^{(n)} \xrightarrow{\sim} B^{(n)} \otimes B^{(m)}$ is given by*

$$R([i]^{(m)} \otimes [j]^{(n)}) = \begin{cases} [i]^{(n)} \otimes [j]^{(m)} & \text{if } i+j \le m,n, \\ [2i+j-m]^{(n)} \otimes [m-i]^{(m)} & \text{if } m \le i+j \le n, \\ [n-j]^{(n)} \otimes [i+2j-n]^{(m)} & \text{if } n \le i+j \le m, \\ [n-m+i]^{(n)} \otimes [j+m-n]^{(m)} & \text{if } i+j \ge m,n. \end{cases} \tag{3.54}$$

The following lemma plays a crucial role in proving our isomorphism theorem.

Lemma 3.10. *Let $\mu = b\Lambda_1 + (n-b)\Lambda_0$. If $m > n$, the following diagram of crystal isomorphisms is commutative.*

$$\begin{tikzcd}
B^{(m-n)} \otimes B(\mu) \arrow[r, "\mathrm{id} \otimes \psi"] \arrow[d, "\Psi"'] & B^{(m-n)} \otimes B(\sigma\mu) \otimes B^{(n)} \arrow[dd, "\Psi \otimes \mathrm{id}"] \\
B(\sigma\mu) \otimes B^{(m)} \arrow[d, "\psi \otimes \mathrm{id}"'] & \\
B(\mu) \otimes B^{(n)} \otimes B^{(m)} & B(\mu) \otimes B^{(m)} \otimes B^{(n)} \arrow[l, "R"']
\end{tikzcd}$$

Proof. Since the crystal $B^{(m-n)} \otimes B(\mu)$ is connected (see [26, 27]), it suffices to check the commutativity for a single element, say $[j]^{(m-n)} \otimes u_\mu \in B^{(m-n)} \otimes B(\mu)$ with $0 \le j \le m-n$. Using (3.48), (3.42), and (3.54), we can show that $[j]^{(m-n)} \otimes u_\mu$ is mapped to $u_\mu \otimes [b]^{(n)} \otimes [j+n-b]^{(m)}$ in both ways. $\square$

By applying the above lemma repeatedly, we obtain:

Corollary 3.11. *For each positive integer $N > 0$, the following diagram of crystal isomorphisms is commutative.*

$$\begin{tikzcd}
B(\lambda) \otimes (B^{(m-n)})^{\otimes N} \otimes B(\mu) \arrow[r, "\psi_\lambda \otimes \psi_\mu"] \arrow[d, "\Psi \circ \cdots \circ \Psi"'] & B(\sigma\lambda) \otimes (B^{(m-n)})^{\otimes (N+1)} \otimes B(\sigma\mu) \otimes B^{(n)} \arrow[dd, "\Psi \circ \cdots \circ \Psi"] \\
B(\lambda) \otimes B(\sigma^N \mu) \otimes (B^{(m)})^{\otimes N} \arrow[d, "\psi_\lambda \otimes \psi_{\sigma^N \mu}"'] & \\
B(\sigma\lambda) \otimes B^{(m-n)} \otimes B(\sigma^{N+1}\mu) \otimes B^{(n)} \otimes (B^{(m)})^{\otimes N} & B(\sigma\lambda) \otimes B^{(m-n)} \otimes B(\sigma^{N+1}\mu) \otimes (B^{(m)})^{\otimes N} \otimes B^{(n)} \arrow[l, "R \circ \cdots \circ R"']
\end{tikzcd}$$

Let $v \otimes w \in B(\lambda) \otimes B(\mu)$. By Theorem 3.7, there exists a positive integer $N > 0$ such that $\psi_\lambda^{(N)}(v) \in u_{\sigma^N \lambda} \otimes (B^{(m-n)})^{\otimes N}$ and $\psi_\mu^{(N)}(v) \in u_{\sigma^N \mu} \otimes (B^{(n)})^{\otimes N}$. Hence we obtain the crystal isomorphism

$$\begin{aligned}
B(\lambda) \otimes B(\mu) &\xrightarrow{\psi_\lambda^{(N)} \otimes \psi_\mu^{(N)}} B(\sigma^N \lambda) \otimes (B^{(m-n)})^{\otimes N} \otimes B(\sigma^N \mu) \otimes (B^{(n)})^{\otimes N} \\
&\xrightarrow{\Psi \circ \cdots \circ \Psi} B(\sigma^N \lambda) \otimes B(\mu) \otimes (B^{(m)})^{\otimes N} \otimes (B^{(n)})^{\otimes N} \\
&\xrightarrow{R \circ \cdots \circ R} B(\sigma^N \lambda) \otimes B(\mu) \otimes (B^{(m)} \otimes B^{(n)})^{\otimes N}
\end{aligned} \tag{3.55}$$

such that $v \otimes w$ is mapped to an element in $u_{\sigma^N \lambda} \otimes u_\mu \otimes (B^{(m)} \otimes B^{(n)})^{\otimes N}$.

Therefore, in order to prove our claim (3.53), it suffices to prove that the following diagram of crystal isomorphisms is commutative.

$$\begin{tikzcd}
B(\lambda) \otimes B(\mu) \arrow[r, "\psi_\lambda^{(N)} \otimes \psi_\mu^{(N)}"] \arrow[d, "\Phi^{(N)}"'] & B(\sigma^N \lambda) \otimes (B^{(m-n)})^{\otimes N} \otimes B(\sigma^N \mu) \otimes (B^{(n)})^{\otimes N} \arrow[d, "\Psi \circ \cdots \circ \Psi"] \\
B(\sigma^N \lambda) \otimes B(\mu) \otimes (B^{(m)} \otimes B^{(n)})^{\otimes N} & B(\sigma^N \lambda) \otimes B(\mu) \otimes (B^{(m)})^{\otimes N} \otimes (B^{(n)})^{\otimes N} \arrow[l, "R \circ \cdots \circ R"']
\end{tikzcd}$$

We will prove our assertion by induction on N. If $N = 1$, there is nothing to prove. Assume that our assertion is true for $N - 1$. Then by the induction hypothesis, the following diagram of crystal isomorphisms is commutative.

$$
\begin{tikzcd}
B(\lambda) \otimes B(\mu) \arrow[r, "\psi_\lambda^{(N-1)} \otimes \psi_\mu^{(N-1)}"] \arrow[d, "\Phi^{(N-1)}"'] & \begin{array}{c} B(\sigma^{N-1}\lambda) \otimes (B^{(m-n)})^{\otimes (N-1)} \\ \otimes B(\sigma^{N-1}\mu) \otimes (B^{(n)})^{\otimes (N-1)} \end{array} \arrow[d, "\Psi \circ \cdots \circ \Psi"] \\
\begin{array}{c} B(\sigma^{N-1}\lambda) \otimes B(\mu) \\ \otimes (B^{(m)} \otimes B^{(n)})^{\otimes (N-1)} \end{array} \arrow[d, "\psi_{\sigma^{N-1}\lambda} \otimes \psi_\mu"'] & \begin{array}{c} B(\sigma^{N-1}\lambda) \otimes B(\mu) \\ \otimes (B^{(m)})^{\otimes (N-1)} \otimes (B^{(n)})^{\otimes (N-1)} \end{array} \arrow[l, "R \circ \cdots \circ R"'] \arrow[d, "\psi_{\sigma^{N-1}\lambda} \otimes \psi_\mu"] \\
\begin{array}{c} B(\sigma^N\lambda) \otimes B^{(m-n)} \otimes B(\sigma\mu) \otimes B^{(n)} \\ \otimes (B^{(m)} \otimes B^{(n)})^{\otimes (N-1)} \end{array} \arrow[d, "\Psi"'] & \begin{array}{c} B(\sigma^N\lambda) \otimes B^{(m-n)} \otimes B(\sigma\mu) \otimes B^{(n)} \\ \otimes (B^{(m)})^{\otimes (N-1)} \otimes (B^{(n)})^{\otimes (N-1)} \end{array} \arrow[l, "R \circ \cdots \circ R"'] \arrow[d, "\Psi"] \\
\begin{array}{c} B(\sigma^N\lambda) \otimes B(\mu) \\ \otimes (B^{(m)} \otimes B^{(n)})^{\otimes N} \end{array} & \begin{array}{c} B(\sigma^N\lambda) \otimes B(\mu) \otimes B^{(m)} \otimes B^{(n)} \\ \otimes (B^{(m)})^{\otimes (N-1)} \otimes (B^{(n)})^{\otimes (N-1)} \end{array} \arrow[l, "R \circ \cdots \circ R"']
\end{tikzcd}
$$

Note that the commutativity of the first square follows from the induction hypothesis and the commutativity of the other squares is trivial.

Next, observe that Corollary 3.11 yields the following commutative diagram of crystal isomorphisms.

$$
\begin{tikzcd}
\begin{array}{c} B(\sigma^{N-1}\lambda) \otimes (B^{(m-n)})^{\otimes (N-1)} \\ \otimes B(\sigma^{N-1}\mu) \otimes (B^{(n)})^{\otimes (N-1)} \end{array} \arrow[r, "\psi_{\sigma^{N-1}\lambda} \otimes \psi_{\sigma^{N-1}\mu}"] \arrow[d, "\Psi \circ \cdots \circ \Psi"'] & \begin{array}{c} B(\sigma^N\lambda) \otimes (B^{(m-n)})^{\otimes N} \\ \otimes B(\sigma^N\mu) \otimes (B^{(n)})^{\otimes N} \end{array} \arrow[dd, "\Psi \circ \cdots \circ \Psi"] \\
\begin{array}{c} B(\sigma^{N-1}\lambda) \otimes B(\mu) \\ \otimes (B^{(m)})^{\otimes (N-1)} \otimes (B^{(n)})^{\otimes (N-1)} \end{array} \arrow[d, "\psi_{\sigma^{N-1}\lambda} \otimes \psi_\mu"'] & \\
\begin{array}{c} B(\sigma^N\lambda) \otimes B^{(m-n)} \otimes B(\sigma\mu) \otimes B^{(n)} \\ \otimes (B^{(m)})^{\otimes (N-1)} \otimes (B^{(n)})^{\otimes (N-1)} \end{array} & \begin{array}{c} B(\sigma^N\lambda) \otimes B^{(m-n)} \otimes B(\sigma\mu) \\ \otimes (B^{(m)})^{\otimes (N-1)} \otimes (B^{(n)})^{\otimes N} \end{array} \arrow[l, "R \circ \cdots \circ R"']
\end{tikzcd}
$$

Moreover, the commutativity of the following diagram is trivial.

$$
\begin{tikzcd}
\begin{array}{c} B(\sigma^N\lambda) \otimes B^{(m-n)} \otimes B(\sigma\mu) \otimes B^{(n)} \\ \otimes (B^{(m)})^{\otimes (N-1)} \otimes (B^{(n)})^{\otimes (N-1)} \end{array} \arrow[d, "\Psi"'] & \begin{array}{c} B(\sigma^N\lambda) \otimes B^{(m-n)} \otimes B(\sigma\mu) \\ \otimes (B^{(m)})^{\otimes (N-1)} \otimes (B^{(n)})^{\otimes N} \end{array} \arrow[l, "R \circ \cdots \circ R"'] \arrow[d, "\Psi"] \\
\begin{array}{c} B(\sigma^N\lambda) \otimes B(\mu) \otimes B^{(m)} \otimes B^{(n)} \\ \otimes (B^{(m)})^{\otimes (N-1)} \otimes (B^{(n)})^{\otimes (N-1)} \end{array} & \begin{array}{c} B(\sigma^N\lambda) \otimes B(\mu) \\ \otimes (B^{(m)})^{\otimes N} \otimes (B^{(n)})^{\otimes N} \end{array} \arrow[l, "R \circ \cdots \circ R"']
\end{tikzcd}
$$

By combining all the commutative diagrams obtained above, we obtain the desired commutative diagram.

To summarise the above discussion, we obtain:

Theorem 3.12. *Suppose $m > n$ and let $\lambda = a\Lambda_1 + (m-n-a)\Lambda_0$ and $\mu = b\Lambda_1 + (n-b)\Lambda_0$ be dominant integral weights of level $m-n$ and n, respectively. Then there exists a crystal isomorphism*

$$B(\lambda) \otimes B(\mu) \xrightarrow{\sim} P_{a,b} \tag{3.56}$$

such that

$$u_\lambda \otimes u_\mu \longmapsto \cdots \otimes [a+b] \otimes [n-b] \otimes [m-n-a+b] \otimes [n-b]. \tag{3.57}$$

4 Level-1 Intertwiners

In this section, we define some $U_q(\widehat{sl}_2)$ intertwiners and study them in the level-1 case. Commutation relations for these operators will be given and some of their matrix elements will be calculated. These will be used in the next section to reduce questions about general level intertwiners to those of level-1 intertwiners.

The reader should recall some notation from Section 2.1. Let σ denote the map exchanging the two fundamental weights $\Lambda_0 \leftrightarrow \Lambda_1$. The set of dominant integral weights of level k will be denoted by P_k^+. For $\lambda \in P_k^+$, $V(\lambda)$ will be the irreducible highest weight module of highest weight λ. Also set $\lambda_\pm = \lambda \pm (\Lambda_1 - \Lambda_0)$ for $\lambda \in P_k^+$. We only consider the cases when $\lambda_\pm \in P_k^+$. The following notation for various $U_q(\widehat{sl}_2)$ intertwiners will be used.

$$\Phi^{(\ell,\ell+k)} \ : \ V_\zeta^{(\ell)} \otimes V(\lambda) \to V(\sigma(\lambda)) \otimes V_\zeta^{(\ell+k)}, \tag{4.1}$$

$$\Phi^{\pm} \ : \ V(\lambda) \to V(\lambda_\pm) \otimes V_\zeta^{(1)}, \tag{4.2}$$

$$\Psi^{*\pm} \ : \ V_\zeta^{(1)} \otimes V(\lambda) \to V(\lambda_\pm). \tag{4.3}$$

As we said in 2.1, we consider the evaluation modules in these formulas as modules of finite rank over the ring of coefficients containing ζ and ζ^{-1}. This implies, in particular, that the intertwiners commute with the multiplication of ζ and ζ^{-1}. We remark also that in each of (4.1)–(4.3) we need completion in the right hand side (see [21] for a detailed discussion).

As to the existence of the intertwiners (4.1), we have the following proposition. The case $\ell = 0$ is given in [21], and the case $k = 1$ in [7]. See also [26] and [27] for the crystal version.

Proposition 4.1. *Let $\lambda, \nu \in P_k^+$. There exists a $U_q'(\widehat{sl}_2)$ intertwiner from $V_\zeta^{(\ell)} \otimes V(\lambda)$ to $V(\nu) \otimes V_\zeta^{(\ell+k)}$ if and only if $\nu = \sigma(\lambda)$. When $\nu = \sigma(\lambda)$, the intertwiner is unique up to scalar multiple.*

Proof. We first prove

$$\begin{aligned}
&\mathrm{Hom}_{U_q'}(V_\zeta^{(\ell)} \otimes V(\lambda), V(\nu) \otimes V_\zeta^{(\ell+k)}) \\
&\qquad \cong W_\lambda^\nu = \{v \in V_\zeta^{(\ell+k)} \otimes V^{(\ell)}_{\zeta q^{-(k+2)/2}}; e_i^{\nu(h_i)+1} v = 0, \mathrm{wt}(v) = \lambda - \nu\}.
\end{aligned}$$

This is very similar to the proof in [7] which is an extended version of the case $\ell = 0$ given in [21]. The steps are

$$\begin{aligned}
\mathrm{Hom}_{U_q'}&(V_\zeta^{(\ell)} \otimes V(\lambda), V(\nu) \otimes V_\zeta^{(\ell+k)}) \\
&\cong \mathrm{Hom}_{U_q'}(V(\lambda), V_\zeta^{(\ell)*a^{-1}} \otimes V(\nu) \otimes V_\zeta^{(\ell+k)}) \\
&\cong \mathrm{Hom}_{U_q'(b^+)}(\mathbf{Q}(q)u_\lambda, V_\zeta^{(\ell)*a^{-1}} \otimes V(\nu) \otimes V_\zeta^{(\ell+k)}) \\
&\cong \mathrm{Hom}_{U_q'(b^+)}(V(\nu)^{*a} \otimes V_\zeta^{(\ell)} \otimes \mathbf{Q}(q)u_\lambda, V_\zeta^{(\ell+k)}) \\
&\cong \mathrm{Hom}_{U_q'(b^+)}(V(\nu)^{*a} \otimes \mathbf{Q}(q)u_\lambda \otimes V^{(\ell)}_{\zeta q^{-k/2}}, V_\zeta^{(\ell+k)}) \\
&\cong \mathrm{Hom}_{U_q'(b^+)}(V(\nu)^{*a} \otimes \mathbf{Q}(q)u_\lambda, V_\zeta^{(\ell+k)} \otimes V^{(\ell)*a}_{\zeta q^{-k/2}}) \\
&\cong \mathrm{Hom}_{U_q'(b^+)}(V(\nu)^{*a} \otimes \mathbf{Q}(q)u_\lambda, V_\zeta^{(\ell+k)} \otimes V^{(\ell)}_{\zeta q^{-(k+2)/2}}) \\
&\cong W_\lambda^\nu.
\end{aligned}$$

Here we used the $U_q'(b^+)$ isomorphism $V_\zeta^{(\ell)} \otimes \mathbf{Q}(q)u_\lambda \cong \mathbf{Q}(q)u_\lambda \otimes V^{(\ell)}_{\zeta q^{-k/2}}$:

$$u_j^{(\ell)} \otimes u_\lambda \mapsto q^{-j(\lambda(h_1)-k/2)} u_\lambda \otimes u_j^{(\ell)}.$$

To complete the proof, we show $\dim(W_\lambda^\nu) = \delta_{\nu,\sigma(\lambda)}$. Suppose $\nu = j\Lambda_1 + (k-j)\Lambda_0$. We solve for the vector $v \in V_\zeta^{(\ell+k)} \otimes V^{(\ell)}_{\zeta q^{-(k+2)/2}}$ which satisfies $e_1^{j+1} v = 0$, $e_0^{k-j+1} v = 0$. It is uniquely given up to constant multiple by

$$y_j^{(k)} = \sum_{i=0}^{\ell} (-1)^i \begin{bmatrix} k+i-j \\ i \end{bmatrix}_q^{\frac{1}{2}} \begin{bmatrix} \ell-i+j \\ j \end{bmatrix}_q^{\frac{1}{2}} u^{(\ell+k)}_{\ell-i+j} \otimes u_i^{(\ell)}.$$

These vectors span a space isomorphic to $V^{(k)}_{\zeta q^{\ell/2}}$. The weight of $y_j^{(k)}$ is $(k-2j)(\Lambda_1 - \Lambda_0)$ and this is equal to $\lambda - \nu$ if and only if $\lambda = \sigma(\nu)$. ☐

Similar existence and uniqueness theorems for the other two intertwiners are also known. We define $|\lambda\rangle$ to be the highest weight vector of $V(\lambda)$ and take $\langle\lambda|$ to be its dual. With these, we normalise the intertwiners as follows:

$$\langle\Lambda_1|\Phi^{(\ell,\ell+1)}|\Lambda_0\rangle(u_\ell^{(\ell)}) = u_{\ell+1}^{(\ell+1)}, \tag{4.4}$$

$$\langle\Lambda_0|\Phi^{(\ell,\ell+1)}|\Lambda_1\rangle(u_0^{(\ell)}) = u_0^{(\ell+1)}, \tag{4.5}$$

$$\langle\lambda_+|\Phi^+|\lambda\rangle = u_1^{(1)}, \qquad \langle\lambda_+|\Psi^{*+}|\lambda\rangle(u_0^{(1)}) = 1, \tag{4.6}$$

$$\langle\lambda_-|\Phi^-|\lambda\rangle = u_0^{(1)}, \qquad \langle\lambda_-|\Psi^{*-}|\lambda\rangle(u_1^{(1)}) = 1. \tag{4.7}$$

Normalisation for the arbitrary level operators $\Phi^{(\ell,\ell')}$ will be given later (page 29, above Proposition 5.3).

The matrix elements of these intertwiners are Laurent polynomials. Therefore, we can write the vertex operators in Laurent series expansions:

$$\Phi_{i,j}^{(\ell,\ell+k)}(\zeta) = \sum_{\kappa} \zeta^{-\kappa}\Phi_{i,j,\kappa}^{(\ell,\ell+k)}, \quad \Phi^{(\ell,\ell+k)}(\zeta)(u_i^{(\ell)} \otimes v) = \sum_{j} \Phi_{i,j}^{(\ell,\ell+k)}(\zeta)v \otimes u_j^{(\ell+k)}, \tag{4.8}$$

$$\Phi_i^{\pm}(\zeta) = \sum_{\kappa} \zeta^{-\kappa}\Phi_{i,\kappa}^{\pm}, \qquad \Phi^{\pm}(\zeta)(v) = \sum_{i=0,1} \Phi_i^{\pm}(\zeta)v \otimes u_i^{(1)}, \tag{4.9}$$

$$\Psi_i^{*\pm}(\zeta) = \sum_{\kappa} \zeta^{-\kappa}\Psi_{i,\kappa}^{*\pm}, \qquad \Psi^{*\pm}(\zeta)(u_i^{(1)} \otimes v) = \Psi_i^{*\pm}(\zeta)v. \tag{4.10}$$

From the normalisations, we see that for Φ_i^{ε}, the sum is taken over κ satisfying $\varepsilon \cdot (-1)^{i+1} = (-1)^{\kappa}$ and that for $\Psi_i^{*\varepsilon}$, it is taken over κ satisfying $\varepsilon \cdot (-1)^{i} = (-1)^{\kappa}$.

Let us state some properties of the level-1 intertwiners. Here, we suppress the appearance of the $\pm$ superscripts on $\Phi^{\pm}(\zeta)$ and $\Psi^{\pm}(\xi)$. Except where we state otherwise, these relations are valid for $\ell \geq 0$, with the identification $\Phi^{(0,1)}(\zeta) = \Phi(\zeta)$.

Proposition 4.2.

$$\begin{aligned}
\xi^{-D}\,\Phi_{s,t}^{(\ell,\ell+1)}(\zeta)\,\xi^{D} &= \Phi_{s,t}^{(\ell,\ell+1)}(\zeta/\xi), && (4.11)\\
\delta_{i+j,\ell} &= g^{(\ell,\ell+1)}\textstyle\sum_{s+t=\ell+1}\Phi_{i,s}^{(\ell,\ell+1)}(-q^{-1}\zeta)\Phi_{j,t}^{(\ell,\ell+1)}(\zeta), && (4.12)\\
\Phi^{(\ell,\ell+1)}(\zeta)\Phi(\xi) &= R^{(1,\ell+1)}(\xi/\zeta)\Phi(\xi)\Phi^{(\ell,\ell+1)}(\zeta), && (4.13)\\
\Phi^{(\ell,\ell+1)}(\zeta)\Psi^{*}(\xi) &= \Psi^{*}(\xi)\Phi^{(\ell,\ell+1)}(\zeta)R^{(\ell,1)}(\zeta/\xi) \quad \textit{for } \ell > 0, && (4.14)\\
\Phi(\zeta)\Psi^{*}(\xi) &= \Psi^{*}(\xi)\Phi(\zeta)\tau(\zeta/\xi), && (4.15)
\end{aligned}$$

$$\begin{aligned}
&\Phi^{(\ell,\ell+1)}(\zeta_1)\Phi^{(\ell',\ell'+1)}(\zeta_2)R^{(\ell',\ell)}(\zeta_2/\zeta_1)\\
&\qquad = R^{(\ell'+1,\ell+1)}(\zeta_2/\zeta_1)\Phi^{(\ell',\ell'+1)}(\zeta_2)\Phi^{(\ell,\ell+1)}(\zeta_1) \quad \textit{for } \ell,\ell' > 0,
\end{aligned} \tag{4.16}$$

where

$$g^{(\ell,\ell')} = \frac{(q^{2\ell+2}\,;\,q^4)_\infty}{(q^{2\ell'+2}\,;\,q^4)_\infty}, \quad \tau(\zeta) = \zeta^{-1}\frac{(q\zeta^2\,;\,q^4)_\infty(q^3\zeta^{-2}\,;\,q^4)_\infty}{(q\zeta^{-2}\,;\,q^4)_\infty(q^3\zeta^{2}\,;\,q^4)_\infty}. \tag{4.17}$$

Proof. All but (4.16) appear in [8]. The last one may be proved by applying (4.13) and (4.14) on the fusion construction of $\Phi^{(\ell,\ell+1)}(\zeta)$ appearing in [7,8]. □

We wish to calculate various level-1, highest weight to highest weight, matrix elements.

Lemma 4.3.

$$\begin{aligned}
&\langle\Lambda_0|\Phi^{(\ell,\ell+1)}(\zeta_1)\Phi^{(\ell',\ell'+1)}(\zeta_2)|\Lambda_0\rangle(u_k^{(\ell)} \otimes u_j^{(\ell')})\\
&\quad = h^{(\ell'+\ell+5)}(\zeta_2/\zeta_1)\Big(\frac{1}{[\ell+1]_q[\ell'+1]_q}\Big)^{\frac{1}{2}}\\
&\qquad \times\Big\{q^{-\frac{1}{2}(\ell'+k-j)}[\ell-k+1]_q^{\frac{1}{2}}[j+1]_q^{\frac{1}{2}}\; u_k^{(\ell+1)} \otimes u_{j+1}^{(\ell'+1)}\\
&\qquad\quad -q^{\frac{1}{2}(\ell-k+j)+1}[\ell'-j+1]_q^{\frac{1}{2}}[k+1]_q^{\frac{1}{2}}(\zeta_2/\zeta_1)\; u_{k+1}^{(\ell+1)} \otimes u_j^{(\ell'+1)}\Big\},
\end{aligned} \tag{4.18}$$

$$\begin{aligned}\langle\Lambda_1|\Phi^{(\ell,\ell+1)}(\zeta_1)\Phi^{(\ell',\ell'+1)}(\zeta_2)|\Lambda_1\rangle(u_k^{(\ell)}\otimes u_j^{(\ell')})\\ = h^{(\ell'+\ell+5)}(\zeta_2/\zeta_1)\Big(\frac{1}{[\ell+1]_q[\ell'+1]_q}\Big)^{\frac12}\\ \times\Big\{-q^{\frac12(\ell'+k-j)+1}[\ell-k+1]_q^{\frac12}[j+1]_q^{\frac12}(\zeta_2/\zeta_1)\,u_k^{(\ell+1)}\otimes u_{j+1}^{(\ell'+1)}\\ +q^{-\frac12(\ell-k+j)}[\ell'-j+1]_q^{\frac12}[k+1]_q^{\frac12}\,u_{k+1}^{(\ell+1)}\otimes u_j^{(\ell'+1)}\Big\},\end{aligned}\tag{4.19}$$

where $h^{(\ell)}(\zeta)=\frac{(q^{\ell+1}\zeta^2;q^4)_\infty}{(q^{\ell-1}\zeta^2;q^4)_\infty}$.

Proof. Let us just show the first one. The second one follows from the first by applying the symmetry σ. We set $F=\langle\Lambda_0|\Phi^{(\ell,\ell+1)}(\zeta_1)\Phi^{(\ell',\ell'+1)}(\zeta_2)|\Lambda_0\rangle$ to simplify notations. This is a map from $V_{\zeta_1}^{(\ell)}\otimes V_{\zeta_2}^{(\ell')}$ to $V_{\zeta_1}^{(\ell+1)}\otimes V_{\zeta_2}^{(\ell'+1)}$. By using $\langle\Lambda_0|e_1=0$, $f_0^2|\Lambda_0\rangle=0$, and the fact that $\Phi^{(\ell,\ell+1)}(\zeta_1)\Phi^{(\ell',\ell'+1)}(\zeta_2)$ is an intertwiner, we can show that

$$F(e_1u)-e_1F(u)=0,\tag{4.20}$$

$$f_0^2F(u)-[2]_qf_0F(f_0u)+F(f_0^2u)=0\tag{4.21}$$

for any $u\in V_{\zeta_1}^{(\ell)}\otimes V_{\zeta_2}^{(\ell')}$. From weight considerations, we know $F(u_0^{(\ell)}\otimes u_0^{(\ell')})$ is a linear combination of $u_0^{(\ell+1)}\otimes u_1^{(\ell'+1)}$ and $u_1^{(\ell+1)}\otimes u_0^{(\ell'+1)}$. The use of (4.20) with $u=u_0^{(\ell)}\otimes u_0^{(\ell')}$ allows us to write

$$\begin{aligned}F(u_0^{(\ell)}\otimes u_0^{(\ell')})=f_{\ell,\ell'}(\zeta_2/\zeta_1)\Big\{\;q^{-\frac12\ell'}[\ell+1]_q^{\frac12}\,u_0^{(\ell+1)}\otimes u_1^{(\ell'+1)}\\ -q^{\frac12\ell+1}[\ell'+1]_q^{\frac12}(\zeta_2/\zeta_1)\,u_1^{(\ell+1)}\otimes u_0^{(\ell'+1)}\Big\}.\end{aligned}$$

Starting from this, and by using (4.20) and (4.21), we may inductively determine F up to the constant multiple $f_{\ell,\ell'}(\zeta_2/\zeta_1)$. To obtain the constant, we use (4.16), which shows

$$\frac{f_{\ell,\ell'}(\zeta_2/\zeta_1)}{f_{\ell',\ell}(\zeta_1/\zeta_2)}=\frac{(q^{\ell+\ell'+6}(\zeta_2/\zeta_1)^2\,;\,q^4)_\infty}{(q^{\ell+\ell'+4}(\zeta_2/\zeta_1)^2\,;\,q^4)_\infty}\frac{(q^{\ell+\ell'+4}(\zeta_1/\zeta_2)^2\,;\,q^4)_\infty}{(q^{\ell+\ell'+6}(\zeta_1/\zeta_2)^2\,;\,q^4)_\infty},$$

together with the normalisation (4.4) and (4.5). □

Lemma 4.4.

$$\begin{aligned}\langle\Lambda_0|\Phi^{(\ell,\ell+1)}(\zeta)\Phi^{+}(\xi)|\Lambda_0\rangle(u_j^{(\ell)})\\ = h^{(\ell+5)}(\xi/\zeta)\Big(\frac{1}{[\ell+1]_q}\Big)^{\frac12}\Big\{\;q^{-\frac12 j}[\ell-j+1]_q^{\frac12}\,u_j^{(\ell+1)}\otimes u_1^{(1)}\\ -\,q^{\frac12(\ell-j)+1}[j+1]_q^{\frac12}(\xi/\zeta)\,u_{j+1}^{(\ell+1)}\otimes u_0^{(1)}\Big\},\end{aligned}\tag{4.22}$$

$$\begin{aligned}\langle\Lambda_1|\Phi^{(\ell,\ell+1)}(\zeta)\Phi^{-}(\xi)|\Lambda_1\rangle(u_j^{(\ell)})\\ = h^{(\ell+5)}(\xi/\zeta)\Big(\frac{1}{[\ell+1]_q}\Big)^{\frac12}\Big\{-\,q^{\frac12 j+1}[\ell-j+1]_q^{\frac12}(\xi/\zeta)\,u_j^{(\ell+1)}\otimes u_1^{(1)}\\ +\,q^{-\frac12(\ell-j)}[j+1]_q^{\frac12}\,u_{j+1}^{(\ell+1)}\otimes u_0^{(1)}\Big\},\end{aligned}\tag{4.23}$$

$$\begin{aligned}&\langle\Lambda_0|\Phi^{-}(\xi)\Phi^{(\ell,\ell+1)}(\zeta)|\Lambda_0\rangle(u_j^{(\ell)})\\&\quad= h^{(\ell+5)}(\zeta/\xi)\Big(\frac{1}{[\ell+1]_q}\Big)^{\frac12}\Big\{\ q^{-\frac12(\ell-j)}[j+1]_q^{\frac12}\ u_0^{(1)}\otimes u_{j+1}^{(\ell+1)}\\&\qquad\qquad - q^{\frac12 j+1}[\ell-j+1]_q^{\frac12}(\zeta/\xi)\ u_1^{(1)}\otimes u_j^{(\ell+1)}\Big\},\end{aligned}\tag{4.24}$$

$$\begin{aligned}&\langle\Lambda_1|\Phi^{+}(\xi)\Phi^{(\ell,\ell+1)}(\zeta)|\Lambda_1\rangle(u_j^{(\ell)})\\&\quad= h^{(\ell+5)}(\zeta/\xi)\Big(\frac{1}{[\ell+1]_q}\Big)^{\frac12}\Big\{-q^{\frac12(\ell-j)+1}[j+1]_q^{\frac12}(\zeta/\xi)\ u_0^{(\ell+1)}\otimes u_{j+1}^{(1)}\\&\qquad\qquad + q^{-\frac12 j}[\ell-j+1]_q^{\frac12}\ u_1^{(1)}\otimes u_j^{(\ell+1)}\Big\}.\end{aligned}\tag{4.25}$$

Proof. Just set ℓ or ℓ' to zero in the preceding Lemma. □

Arguments similar to the proof of Lemma 4.3 show:

Lemma 4.5.

$$\langle\Lambda_0|\Phi^{(\ell,\ell+1)}(\zeta)\Psi^{*+}(\xi)|\Lambda_0\rangle(u_j^{(\ell)}\otimes u_0^{(1)}) = h^{(\ell+2)}(\xi/\zeta)\big(\tfrac{[\ell+1-j]_q}{[\ell+1]_q}\big)^{\frac12}q^{-\frac12 j}u_j^{(\ell+1)},\tag{4.26}$$

$$\langle\Lambda_0|\Phi^{(\ell,\ell+1)}(\zeta)\Psi^{*+}(\xi)|\Lambda_0\rangle(u_j^{(\ell)}\otimes u_1^{(1)}) = h^{(\ell+2)}(\xi/\zeta)\big(\tfrac{[j+1]_q}{[\ell+1]_q}\big)^{\frac12}q^{\frac12(\ell-j)}(\xi/\zeta)u_{j+1}^{(\ell+1)},\tag{4.27}$$

$$\langle\Lambda_1|\Phi^{(\ell,\ell+1)}(\zeta)\Psi^{*-}(\xi)|\Lambda_1\rangle(u_j^{(\ell)}\otimes u_0^{(1)}) = h^{(\ell+2)}(\xi/\zeta)\big(\tfrac{[\ell+1-j]_q}{[\ell+1]_q}\big)^{\frac12}q^{\frac12 j}(\xi/\zeta)u_j^{(\ell+1)},\tag{4.28}$$

$$\langle\Lambda_1|\Phi^{(\ell,\ell+1)}(\zeta)\Psi^{*-}(\xi)|\Lambda_1\rangle(u_j^{(\ell)}\otimes u_1^{(1)}) = h^{(\ell+2)}(\xi/\zeta)\big(\tfrac{[j+1]_q}{[\ell+1]_q}\big)^{\frac12}q^{-\frac12(\ell-j)}u_{j+1}^{(\ell+1)},\tag{4.29}$$

$$\langle\Lambda_0|\Psi^{*-}(\xi)\Phi^{(\ell,\ell+1)}(\zeta)|\Lambda_0\rangle(u_0^{(1)}\otimes u_j^{(\ell)}) = h^{(\ell+2)}(\zeta/\xi)\big(\tfrac{[\ell+1-j]_q}{[\ell+1]_q}\big)^{\frac12}q^{\frac12 j}(\zeta/\xi)u_j^{(\ell+1)},\tag{4.30}$$

$$\langle\Lambda_0|\Psi^{*-}(\xi)\Phi^{(\ell,\ell+1)}(\zeta)|\Lambda_0\rangle(u_1^{(1)}\otimes u_j^{(\ell)}) = h^{(\ell+2)}(\zeta/\xi)\big(\tfrac{[j+1]_q}{[\ell+1]_q}\big)^{\frac12}q^{-\frac12(\ell-j)}u_{j+1}^{(\ell+1)},\tag{4.31}$$

$$\langle\Lambda_1|\Psi^{*+}(\xi)\Phi^{(\ell,\ell+1)}(\zeta)|\Lambda_1\rangle(u_0^{(1)}\otimes u_j^{(\ell)}) = h^{(\ell+2)}(\zeta/\xi)\big(\tfrac{[\ell+1-j]_q}{[\ell+1]_q}\big)^{\frac12}q^{-\frac12 j}u_j^{(\ell+1)},\tag{4.32}$$

$$\langle\Lambda_1|\Psi^{*+}(\xi)\Phi^{(\ell,\ell+1)}(\zeta)|\Lambda_1\rangle(u_1^{(1)}\otimes u_j^{(\ell)}) = h^{(\ell+2)}(\zeta/\xi)\big(\tfrac{[j+1]_q}{[\ell+1]_q}\big)^{\frac12}q^{\frac12(\ell-j)}(\zeta/\xi)u_{j+1}^{(\ell+1)}.\tag{4.33}$$

5 Commutativity with the DVA Action

In this section, we show that $\Phi^{(\ell',\ell'+1)}(\zeta)\Phi^{(\ell,\ell')}(\zeta)$ commutes with the DVA action on $V(\lambda)\otimes V(\Lambda_i)$. This will allow us to reduce questions about general level intertwiners to those of level-1 intertwiners. Results on general level intertwiners will be used in Section 6.3 to diagonalise the transfer matrix.

Let $\lambda\in P_k^+$ and define $\psi^{(\lambda,i,\pm)}(\xi)$ to be the composition of operators given by:

$$V(\lambda)\otimes V(\Lambda_i)\xrightarrow{\Phi^{\pm}(\xi)\otimes\mathrm{id}} V(\lambda_\pm)\otimes V_\xi^{(1)}\otimes V(\Lambda_i)\xrightarrow{\mathrm{id}\otimes\Psi^*(\xi)} V(\lambda_\pm)\otimes V(\Lambda_{1-i}).\tag{5.1}$$

This is equivalent to defining

$$\psi^{(\lambda,i,\pm)}(\xi)=\sum_\kappa \psi_\kappa^{(\lambda,i,\pm)}\xi^{-\kappa}=\sum_{j=0,1}\Phi_j^{\pm}(\xi)\otimes\Psi_j^*(\xi).\tag{5.2}$$

Each component

$$\psi_\kappa^{(\lambda,i,\pm)} : V(\lambda) \otimes V(\Lambda_i) \to V(\lambda_\pm) \otimes V(\Lambda_{1-i}) \tag{5.3}$$

is a $U_q'(\widehat{sl}_2)$ homomorphism. In [15], Jimbo and Shiraishi considered the irreducible decomposition

$$V(\lambda) \otimes V(\Lambda_i) = \bigoplus_\nu V(\nu) \otimes \Omega_\nu^{\lambda,\Lambda_i}, \tag{5.4}$$

and constructed an action of the deformed Virasoro algebra on $\Omega_\nu^{\lambda,\Lambda_i}$ by using the operator $\psi^{(\lambda,i,\pm)}(\xi)$.

Now, define $\phi^{(\ell,\lambda,i)}(\zeta)$ to be the composition of operators given by:

$$\begin{aligned} V_\zeta^{(\ell)} \otimes V(\lambda) \otimes V(\Lambda_i) &\xrightarrow{\Phi^{(\ell,\ell+k)}(\zeta)\otimes \mathrm{id}} V(\sigma(\lambda)) \otimes V_\zeta^{(\ell+k)} \otimes V(\Lambda_i) \\ &\xrightarrow{\mathrm{id}\otimes \Phi^{(\ell+k,\ell+k+1)}(\zeta)} V(\sigma(\lambda)) \otimes V(\Lambda_{1-i}) \otimes V_\zeta^{(\ell+k+1)}. \end{aligned}$$

We shall show

$$\phi^{(\ell,\lambda_\pm,1-i)}(\zeta) \circ \left(\mathrm{id} \otimes \psi^{(\lambda,i,\pm)}(\xi)\right) = \left(\mathrm{id} \otimes \psi^{(\sigma(\lambda),1-i,\mp)}(\xi)\right) \circ \phi^{(\ell,\lambda,i)}(\zeta). \tag{5.5}$$

This will imply the commutativity of $\phi^{(\ell,\lambda,i)}$ with the DVA action mentioned above.

Let us state a small lemma before considering the level-1 case.

Lemma 5.1. *Fix any $U_q'(\widehat{sl}_2)$ modules V and W. Let $\Theta : V(\lambda) \otimes V \to W \otimes V(\mu)$ be a $U_q'(\widehat{sl}_2)$ intertwiner. Then any matrix element of Θ (as an operator in $\mathrm{End}(V,W)$) may be written in the form*

$$\sum \left(\ \cdots \ \right) \circ \langle \mu | \Theta | \lambda \rangle \circ \left(\ \cdots \ \right), \tag{5.6}$$

where the parentheses signify appropriate $U_q'(\widehat{sl}_2)$ actions on V and W, respectively.

Proof. The proof follows from the simple fact that Θ is an intertwiner. □

Now we show (5.5) in the level-1 case.

Proposition 5.2. *The equality,*

$$\phi^{(\ell,\Lambda_{1-j},1-i)}(\zeta) \circ \left(\mathrm{id} \otimes \psi^{(\Lambda_j,i,\pm)}(\xi)\right) = \left(\mathrm{id} \otimes \psi^{(\Lambda_{1-j},1-i,\mp)}(\xi)\right) \circ \phi^{(\ell,\Lambda_j,i)}(\zeta) \tag{5.7}$$

holds for their matrix elements as Laurent series of ζ and ξ. They contain no poles.

Proof. With the help of equation (4.13) and (4.14) we can show:

$$\begin{aligned} &\left(\Phi^{(\ell+1,\ell+2)}(\zeta)\Phi^{(\ell,\ell+1)}(\zeta)\right) \circ \left(\Psi^*(\xi)\Phi(\xi)\right) \\ &\qquad = \ \Phi^{(\ell+1,\ell+2)}(\zeta)\Psi^*(\xi)\Phi^{(\ell,\ell+1)}(\zeta)\Phi(\xi) \end{aligned}$$

$$
\begin{aligned}
&= \Phi^{(\ell+1,\ell+2)}(\zeta)\Psi^*(\xi)R^{(1,\ell+1)}(\xi/\zeta)\Phi(\xi)\Phi^{(\ell,\ell+1)}(\zeta) \\
&= \Psi^*(\xi)\Phi^{(\ell+1,\ell+2)}(\zeta)\Phi(\xi)\Phi^{(\ell,\ell+1)}(\zeta) \\
&= \big(\Psi^*(\xi)\Phi(\xi)\big)\circ\big(\Phi^{(\ell+1,\ell+2)}(\zeta)\Phi^{(\ell,\ell+1)}(\zeta)\big).
\end{aligned}
$$

So the two sides agree as meromorphic functions. Let us look at the structure of poles. We have

$$
\begin{aligned}
&\langle\Lambda_j|\otimes\langle\Lambda_i|\big(\Phi^{(\ell+1,\ell+2)}(\zeta)\Phi^{(\ell,\ell+1)}(\zeta)\big)\circ\big(\Psi^*(\xi)\Phi(\xi)\big)|\Lambda_j\rangle\otimes|\Lambda_i\rangle \\
&\qquad = \langle\Lambda_i|\Phi^{(\ell+1,\ell+2)}(\zeta)\Psi^*(\xi)|\Lambda_i\rangle\circ\langle\Lambda_j|\Phi^{(\ell,\ell+1)}(\zeta)\Phi(\xi)|\Lambda_j\rangle.
\end{aligned}
$$

Using the equations (4.22), (4.23) and also equations (4.26) to (4.29) with ℓ replaced by $\ell+1$, we see that a pole can occur in the above only if $1-q^{\ell+2}(\xi/\zeta)^2=0$. If ζ and ξ satisfy this relation, there exists a submodule isomorphic to some $V_\mu^{(\ell)}$ lying inside $V_\zeta^{(\ell+1)}\otimes V_\xi^{(1)}$. When $\xi=\zeta q^{-\frac{1}{2}(\ell+2)}$, a submodule isomorphic to $V^{(\ell)}_{\zeta q^{\frac{1}{2}}}$ lying inside $V_\zeta^{(\ell+1)}\otimes V_\xi^{(1)}$ is spanned by

$$
\Big\{[\ell+1-k]_q^{\frac{1}{2}}\ u_k^{(\ell+1)}\otimes u_1^{(1)}-[k+1]_q^{\frac{1}{2}}\ u_{k+1}^{(\ell+1)}\otimes u_0^{(1)}\Big\}_{k=0}^{\ell}. \tag{5.8}
$$

Again, from the same set of equations, we see that the image of $\langle\Lambda_j|\Phi^{(\ell,\ell+1)}(\zeta)\Phi(\xi)|\Lambda_j\rangle$ lies inside this submodule. We also see that $\langle\Lambda_i|\Phi^{(\ell+1,\ell+2)}(\zeta)\Psi^*(\xi)|\Lambda_i\rangle$ sends this submodule to zero. Therefore, the above matrix element contains no pole.

In view of Lemma 5.1, the general matrix element can be written in the form

$$
\sum\ \big(\cdots\big)\circ\langle\Lambda_i|\Phi^{(\ell+1,\ell+2)}(\zeta)\Psi^*(\xi)|\Lambda_i\rangle\circ\big(\cdots\big)\circ\langle\Lambda_j|\Phi^{(\ell,\ell+1)}(\zeta)\Phi(\xi)|\Lambda_j\rangle\circ\big(\cdots\big)\,, \tag{5.9}
$$

where the parentheses are to be filled with $U_q'(\widehat{sl}_2)$ actions. As the action of $U_q'(\widehat{sl}_2)$ cannot produce additional poles, the only possible poles will occur at $1-q^{\ell+2}(\xi/\zeta)^2=0$. Again, the image of the first map will lie inside some submodule $V_\mu^{(\ell)}$. The $U_q'(\widehat{sl}_2)$-action will still preserve this submodule. Then the second map will send this submodule to zero. This shows that the general matrix element also contains no pole. □

We now show that the commutativity with the DVA action allows us to construct $\Phi^{(\ell,\ell+k)}$ from lower level operators. Assume from now on that equation (5.5) is true as Laurent series for $\lambda\in P_{k-1}^+$. Now, $\Phi^{(\ell+k-1,\ell+k)}(\zeta)\Phi^{(\ell,\ell+k-1)}(\zeta)$ is a map from

$$
V_\zeta^{(\ell)}\otimes V(\lambda)\otimes V(\Lambda_i)=\bigoplus_\nu V_\zeta^{(\ell)}\otimes V(\nu)\otimes\Omega_\nu^{\lambda,\Lambda_i}, \tag{5.10}
$$

where the sum runs over all level-k weights, to the space

$$
V(\sigma(\lambda))\otimes V(\Lambda_{1-i})\otimes V_\zeta^{(\ell+k)}=\bigoplus_\nu V(\sigma(\nu))\otimes V_\zeta^{(\ell+k)}\otimes\Omega_\nu^{\lambda,\Lambda_i}. \tag{5.11}
$$

Here, we have used the $\mathbf{Z}_2$-symmetry to write $\Omega_{\sigma(\nu)}^{\sigma(\lambda),\Lambda_{1-i}}=\Omega_\nu^{\lambda,\Lambda_i}$. Recalling Proposition 4.1, we may write

$$
\Phi^{(\ell+k-1,\ell+k)}(\zeta)\Phi^{(\ell,\ell+k-1)}(\zeta)=\bigoplus_\nu \Phi_\nu^{(\ell,\ell+k)}(\zeta)\otimes\Xi_\nu^{\lambda,\Lambda_i}. \tag{5.12}
$$

The subscript ν in $\Phi_\nu^{(\ell,\ell+k)}(\zeta)$ has been added to show which space it acts on. Now, each $\Omega_\nu^{\lambda,\Lambda_i}$ is irreducible. Hence the commutativity with the DVA action shows that each $\Xi_\nu^{\lambda,\Lambda_i}$ is a constant map. We normalise the higher level intertwiner $\Phi_\nu^{(\ell,\ell+k)}(\zeta)$ so that this constant is equal to 1 for the highest component, i.e., for $\nu = \lambda + \Lambda_i$. This normalisation is independent of the way we break up the level-k weight into level-1 weights, as can be seen by the use of

$$\langle\Lambda_1|\Phi^{(\ell,\ell+1)}(\zeta)|\Lambda_0\rangle(u_j^{(\ell)}) \;=\; \left(q^{j-\ell}\frac{[j+1]}{[\ell+1]}\right)^{\frac{1}{2}} u_{j+1}^{(\ell+1)}, \tag{5.13}$$

$$\langle\Lambda_0|\Phi^{(\ell,\ell+1)}(\zeta)|\Lambda_1\rangle(u_j^{(\ell)}) \;=\; \left(q^{-j}\frac{[\ell-j+1]}{[\ell+1]}\right)^{\frac{1}{2}} u_j^{(\ell+1)}. \tag{5.14}$$

The next proposition is more of a definition.

Proposition 5.3. *The map $\Phi^{(\ell+k-1,\ell+k)}(\zeta)\Phi^{(\ell,\ell+k-1)}(\zeta)$ restricted to $V_\zeta^{(\ell)}\otimes V(\lambda+\Lambda_i)\otimes\Omega_{\lambda+\Lambda_i}^{\lambda,\Lambda_i}$ is equal to $\Phi_{\lambda+\Lambda_i}^{(\ell,\ell+k)}(\zeta)\otimes \mathrm{id}$.*

Later on, we will say something about the coefficients of other components. But for now, let us continue with proving the commutativity with the DVA action. Since we now know how to construct $\Phi^{(\ell,\ell+k)}(\zeta)$ from lower level operators, we can find its commutation relations. Except where we state otherwise, the relations we give in the following proposition are valid for $\ell \geq 0$.

Proposition 5.4.

$$\xi^{-D}\,\Phi_{s,t}^{(\ell,\ell')}(\zeta)\,\xi^D \;=\; \Phi_{s,t}^{(\ell,\ell')}(\zeta/\xi), \tag{5.15}$$

$$\delta_{i+j,\ell} \;=\; g^{(\ell,\ell')}\textstyle\sum_{s+t=\ell'}\Phi_{i,s}^{(\ell,\ell')}(-q^{-1}\zeta)\Phi_{j,t}^{(\ell,\ell')}(\zeta), \tag{5.16}$$

$$\Phi^{(\ell,\ell')}(\zeta)\Phi^{\pm}(\xi) \;=\; R^{(1,\ell')}(\xi/\zeta)\Phi^{\mp}(\xi)\Phi^{(\ell,\ell')}(\zeta), \tag{5.17}$$

$$\Phi^{(\ell,\ell')}(\zeta)\Psi^{*\pm}(\xi) \;=\; \Psi^{*\mp}(\xi)\Phi^{(\ell,\ell')}(\zeta)R^{(\ell,1)}(\zeta/\xi) \quad \textit{for } \ell>0, \tag{5.18}$$

$$\Phi^{(0,\ell)}(\zeta)\Psi^{*\pm}(\xi) \;=\; \Psi^{*\mp}(\xi)\Phi^{(0,\ell)}(\zeta)\tau(\zeta/\xi). \tag{5.19}$$

Proof. Let us prove equation (5.17) as an example. We will consider just one set of the signs involved. First consider the map

$$\mathrm{id}\otimes\Phi^+(\xi) \;:\; V(\lambda)\otimes V(\Lambda_0)\to V(\lambda)\otimes V(\Lambda_1)\otimes V_\xi^{(1)}. \tag{5.20}$$

As before, we may write this map as

$$\mathrm{id}\otimes\Phi^+(\xi) \;=\; \Big(\bigoplus_\nu \Phi_\nu^+(\xi)\otimes\Xi_\nu^+\Big)+\Big(\bigoplus_\nu \Phi_\nu^-(\xi)\otimes\Xi_\nu^-\Big), \tag{5.21}$$

where the maps on the right hand side are from $V(\nu)\otimes\Omega_\nu^{\lambda,\Lambda_0}$ to $V(\nu_\pm)\otimes V_\xi^{(1)}\otimes\Omega_{\nu_\pm}^{\lambda,\Lambda_1}$. As in the proof of Proposition 5.3 we take the highest weight matrix element and apply both sides on $u_0^{(\ell)}$ to conclude $\Xi_{\lambda+\Lambda_0}^+$ is nonzero. By $\mathbf{Z}_2$-symmetry, the map

$$\mathrm{id}\otimes\Phi^-(\xi) \;:\; V(\sigma(\lambda))\otimes V(\Lambda_1)\to V(\sigma(\lambda))\otimes V(\Lambda_0)\otimes V_\xi^{(1)} \tag{5.22}$$

breaks up as

$$\mathrm{id} \otimes \Phi^-(\xi) \;=\; \Big(\bigoplus_{\nu} \Phi^-_{\sigma(\nu)}(\xi) \otimes \Xi^+_\nu\Big) + \Big(\bigoplus_{\nu} \Phi^+_{\sigma(\nu)}(\xi) \otimes \Xi^-_\nu\Big). \tag{5.23}$$

With this much in hand, we proceed by induction. By equation (4.13) we have,

$$\begin{aligned} &\Phi^{(\ell+k-1,\ell+k)}(\zeta)\,\Phi^{(\ell,\ell+k-1)}(\zeta)(\mathrm{id}\otimes\Phi^+(\xi)) && (5.24)\\ &\quad= \Phi^{(\ell+k-1,\ell+k)}(\zeta)\big(\mathrm{id}\otimes\Phi^+(\xi)\big)\,\Phi^{(\ell,\ell+k-1)}(\zeta) && (5.25)\\ &\quad= R^{(1,\ell+k)}(\xi/\zeta)\big(\mathrm{id}\otimes\Phi^-(\xi)\big)\,\Phi^{(\ell+k-1,\ell+k)}(\zeta)\,\Phi^{(\ell,\ell+k-1)}(\zeta). && (5.26)\end{aligned}$$

If we substitute equations (5.12), (5.21), and (5.23) into both sides and pick up the term initiating at $V(\lambda+\Lambda_0)\otimes\Omega^{\lambda,\Lambda_0}_{\lambda+\Lambda_0}$ and terminating at $V(\sigma(\lambda+\Lambda_1))\otimes\Omega^{\lambda,\Lambda_1}_{\lambda+\Lambda_1}$, and apply Proposition 5.3 to the outcome, we get:

$$\big(\Phi^{(\ell,\ell+k)}(\zeta)\Phi^+_{\lambda+\Lambda_0}(\xi)\big)\otimes\Xi^+_{\lambda+\Lambda_0} = \big(R^{(1,\ell+k)}(\xi/\zeta)\Phi^-_{\sigma(\lambda+\Lambda_0)}(\xi)\Phi^{(\ell,\ell+k)}(\zeta)\big)\otimes\Xi^+_{\lambda+\Lambda_0}. \tag{5.27}$$

We already know $\Xi^+_{\lambda+\Lambda_0}$ is nonzero, so dividing them out, we have the result. □

In much the same way, we can also calculate the higher level matrix elements. Here we only write down what is needed in proving the commutativity with the DVA action.

Lemma 5.5. *Let λ be of level $k-1$. Then*

$$\begin{aligned} &\langle\sigma(\lambda)+\Lambda_0|\Phi^{(\ell,\ell+k)}(\zeta)\Phi^+(\xi)|\lambda+\Lambda_0\rangle\\ &\qquad= \langle\Lambda_0|\Phi^{(\ell+k-1,\ell+k)}(\zeta)\Phi^+|\Lambda_0\rangle\circ\langle\sigma(\lambda)|\Phi^{(\ell,\ell+k-1)}(\zeta)|\lambda\rangle,\\ &\langle\sigma(\lambda)+\Lambda_1|\Phi^{(\ell,\ell+k)}(\zeta)\Phi^-(\xi)|\lambda+\Lambda_1\rangle\\ &\qquad= \langle\Lambda_1|\Phi^{(\ell+k-1,\ell+k)}(\zeta)\Phi^-|\Lambda_1\rangle\circ\langle\sigma(\lambda)|\Phi^{(\ell,\ell+k-1)}(\zeta)|\lambda\rangle.\end{aligned}$$

We are now ready for the induction step in proving commutativity with the DVA action.

Theorem 5.6. *The equality,*

$$\phi^{(\ell,\lambda_\pm,1-i)}(\zeta)\circ\big(\mathrm{id}\otimes\psi^{(\lambda,i,\pm)}(\xi)\big) \;=\; \big(\mathrm{id}\otimes\psi^{(\sigma(\lambda),1-i,\mp)}(\xi)\big)\circ\phi^{(\ell,\lambda,i)}(\zeta), \tag{5.28}$$

holds for their matrix elements as Laurent series of ζ and ξ. They contain no poles.

Proof. We are assuming that the statement is true for levels less than k. Hence Proposition 5.3, Proposition 5.4 and Lemma 5.5 hold true for level k. Applying (5.17) and then (4.14), we show the equality of both sides as meromorphic functions.

$$\begin{aligned}\mathrm{LHS} &= \Phi^{(\ell+k,\ell+k+1)}(\zeta)\Psi^*(\xi)\Phi^{(\ell,\ell+k)}(\zeta)\Phi(\xi)\\ &= \Phi^{(\ell+k,\ell+k+1)}(\zeta)\Psi^*(\xi)R^{(1,\ell+k)}(\xi/\zeta)\Phi(\xi)\Phi^{(\ell,\ell+k)}(\zeta)\\ &= \Psi^*(\xi)\Phi^{(\ell+k,\ell+k+1)}(\zeta)\Phi(\xi)\Phi^{(\ell,\ell+k)}(\zeta)\\ &= \mathrm{RHS}.\end{aligned}$$

For the rest of the proof, we will explain the case when the left hand side contains the $+$ sign. The other case can be taken care of similarly. For this case, we may write $\lambda = \lambda' + \Lambda_0$ with λ' of level $k-1$. We use Lemma 5.5 to show:

$$\begin{aligned}
&\langle\sigma(\lambda_+)| \otimes \langle\Lambda_i|\big(\Phi^{(\ell+k,\ell+k+1)}(\zeta)\Phi^{(\ell,\ell+k)}(\zeta)\big) \circ \big(\Psi^*(\xi)\Phi(\xi)\big)|\lambda\rangle \otimes |\Lambda_i\rangle \\
&\quad = \langle\Lambda_i|\Phi^{(\ell+k,\ell+k+1)}(\zeta)\Psi^*(\xi)|\Lambda_i\rangle \circ \langle\sigma(\lambda') + \Lambda_0|\Phi^{(\ell,\ell+k)}(\zeta)\Phi(\xi)|\lambda' + \Lambda_0\rangle \\
&\quad = \langle\Lambda_i|\Phi^{(\ell+k,\ell+k+1)}(\zeta)\Psi^*(\xi)|\Lambda_i\rangle \\
&\qquad\qquad \circ \langle\Lambda_0|\Phi^{(\ell+k,\ell+k+1)}(\zeta)\Phi^{+}(\xi)|\Lambda_0\rangle \circ \langle\sigma(\lambda')|\Phi^{(\ell,\ell+k)}(\zeta)|\lambda'\rangle.
\end{aligned}$$

Now, we may argue as in the proof of Proposition 5.2 to show that the above identity contains no pole. $\square$

We remark on the coefficients of the components of $\Phi^{(\ell+k-1,\ell+k)}(\zeta)\Phi^{(\ell,\ell+k)}(\zeta)$ before closing this section.

Proposition 5.7.

$$\Phi^{(\ell+k-1,\ell+k)}(\zeta)\Phi^{(\ell,\ell+k-1)}(\zeta) = \bigoplus_{\nu} c_\nu \cdot \Phi_{\nu}^{(\ell,\ell+k)}(\zeta) \otimes \mathrm{id}, \tag{5.29}$$

with each $(c_\nu)^2 = 1$.

Proof. We have only to show $(c_\nu)^2 = 1$. Using equation (5.16), we have

$$\begin{aligned}
g^{(\ell,\ell+k)} \bigoplus_{\nu} \Big\{ \sum_{s+t=\ell+k} (c_\nu)^2 \cdot \Phi^{(\ell,\ell+k)}_{\nu,\ i,s}(-q^{-1}\zeta)\Phi^{(\ell,\ell+k)}_{\nu,\ j,t}(\zeta) \otimes \mathrm{id} \Big\} && (5.30)\\
= \bigoplus_{\nu} \Big\{ (c_\nu)^2 \cdot \mathrm{id}_{V(\nu)} \otimes \mathrm{id} \Big\}\, \delta_{i+j,\ell}. && (5.31)
\end{aligned}$$

If we calculate the same thing with the left hand side expression of equation (5.29), we get

$$\bigoplus_{\nu} \Big\{ \mathrm{id}_{V(\nu)} \otimes \mathrm{id} \Big\}\, \delta_{i+j,\ell}. \tag{5.32}$$

This shows that each $(c_\nu)^2 = 1$. $\square$

6 Diagonalisation of the Transfer Matrix

In this section, we identify the space of states, and the half and full transfer matrices of the alternating spin vertex model in terms of the representation theory of $U_q(\widehat{sl}_2)$. We diagonalise the full transfer matrix in terms of the spin-0 and spin-$\frac{1}{2}$ states mentioned in the introduction, and compute two-particle S-matrix elements.

6.1 The Space of States

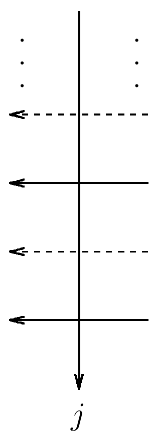
In Section 3, we have shown that there is a crystal isomorphism $P_{a,b} \simeq B(\lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)})$. This leads us to conjecture that we can extend this isomorphism away from $q = 0$, and identify the space of eigenstates of the corner transfer matrix $A_{NW}(\zeta)$ with $\mathcal{H} \equiv \bigoplus_{a,b} \mathcal{H}_{a,b}$, where $\mathcal{H}_{a,b} = V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)})$, and $0 \le a \le m-n$, $0 \le b \le n$. The operator $A_{NW}(\zeta)$ will act as $\mathcal{H}_{a,b} \to \mathcal{H}_{a,b}$. Then $\mathcal{F} \equiv \mathcal{H} \otimes \mathcal{H}^*$ will be the space on which our full transfer matrix acts. Here, $\mathcal{H}^*$ is the dual space, defined using the $U_q'(\widehat{sl}_2)$ anti-automorphism b given in [28]. The motivation for this definition, and the reason for the use of this particular anti-automorphism are discussed in the similar context of the pure spin-$\frac{1}{2}$ vertex model in [3].

We can identify an element $\langle f| \in \mathcal{F}^*$ with an element $|f\rangle \in \mathcal{F}$ via the pairing $\langle f|g\rangle = \mathrm{Tr}_{\mathcal{H}}(f \circ g)$. Here, we have used the canonical isomorphism $\mathcal{F} \simeq \mathrm{End}(\mathcal{H})$ to identify $f, g \in \mathcal{F}$ as elements of $\mathrm{End}(\mathcal{H})$ in the trace formula.

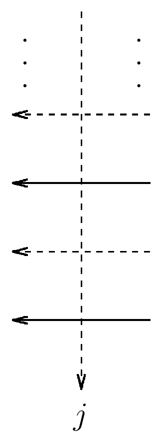
6.2 Half and full transfer matrices

A half transfer matrix represents the insertion of a half-infinite column of lattice vertices. There are two types of half transfer matrices for the alternating spin model - those associated with the insertions of columns with spin-$\frac{n}{2}$ and spin-$\frac{m}{2}$ vertical lines. These are shown in Figures 3 (a) and (b) respectively.

Figure 3 (a)



(b)



As lattice insertions these will be the maps $\mathcal{H}_{a,b} \to \mathcal{H}_{a,n-b}$ and $\mathcal{H}_{a,b} \to \mathcal{H}_{m-n-a,n-b}$ respectively (one can see this by an inspection of the ground state configuration shown in Figure 2).

As discussed in the introduction, we identify these lattice insertions with components

$\phi^A_j(\zeta)$ and $\phi^B_j(\zeta)$ of the following intertwiners.

$$\begin{aligned}
\phi^A(\zeta) \ &: \ V(\lambda_a^{(m-n)})\otimes V(\lambda_b^{(n)}) \xrightarrow{\mathrm{id}\otimes\Phi^{(0,n)}(\zeta)} && V(\lambda_a^{(m-n)})\otimes V(\sigma(\lambda_b^{(n)}))\otimes V_\zeta^{(n)},\\
\phi^B(\zeta) \ &: \ V(\lambda_a^{(m-n)})\otimes V(\lambda_b^{(n)}) \xrightarrow{\Phi^{(0,m-n)}(\zeta)\otimes\mathrm{id}} && V(\sigma(\lambda_a^{(m-n)}))\otimes V_\zeta^{(m-n)}\otimes V(\lambda_b^{(n)})\\
&\qquad\qquad\qquad\qquad\qquad \xrightarrow{\mathrm{id}\otimes\Phi^{(m-n,m)}(\zeta)} && V(\sigma(\lambda_a^{(m-n)}))\otimes V(\sigma(\lambda_b^{(n)}))\otimes V_\zeta^{(m)}.
\end{aligned}$$

Here, $\Phi^{(k,l)}(\zeta)$ is the perfect intertwiner defined in Section 4. If $v\otimes v'\in V(\lambda_a^{(m-n)})\otimes V(\lambda_b^{(n)})$, then the components $\phi^A_j(\zeta)$ and $\phi^B_j(\zeta)$ are defined by

$$\begin{aligned}
\phi^A(v\otimes v') &= \sum_{j=0}^{n}\phi^A_j(v\otimes v')\otimes u_j^{(n)}, \quad & \phi^A_j(v\otimes v') &= v\otimes\Phi_j^{(0,n)}v',\\
\phi^B(v\otimes v') &= \sum_{j=0}^{m}\phi^B_j(v\otimes v')\otimes u_j^{(m)}, \quad & \phi^B_j(v\otimes v') &= \sum_{j'=0}^{m-n}\Phi_{j'}^{(0,m-n)}v\otimes\Phi_{j',j}^{(m-n,m)}v'.
\end{aligned} \tag{6.1}$$

Here, for clarity, we have suppressed the ζ dependence of all our intertwiners.

Now consider the corresponding full transfer matrices, i.e., those associated with the insertion of full, double-infinite, columns of lattice vertices. Again there will be two such transfer matrices, $T^A(\zeta)$ and $T^B(\zeta)$, associated with spin-$\frac{n}{2}$ and spin-$\frac{m}{2}$ auxiliary spaces respectively. We identify these in terms of intertwiners that act on the tensor product space $\mathcal{H}_{a,b}\otimes\mathcal{H}^*_{a',b'}$ as follows

$$\mathcal{H}_{a,b}\otimes\mathcal{H}^*_{a',b'} \xrightarrow{\phi^A(\zeta)\otimes\mathrm{id}} \mathcal{H}_{a,n-b}\otimes V_\zeta^{(n)}\otimes\mathcal{H}^*_{a',b'} \xrightarrow{\mathrm{id}\otimes\phi^A(\zeta)^t} \mathcal{H}_{a,n-b}\otimes\mathcal{H}^*_{a',n-b'} \tag{6.2}$$

$$\begin{aligned}
\mathcal{H}_{a,b}\otimes\mathcal{H}^*_{a',b'} &\xrightarrow{\phi^B(\zeta)\otimes\mathrm{id}} \mathcal{H}_{m-n-a,n-b}\otimes V_\zeta^{(m)}\otimes\mathcal{H}^*_{a',b'}\\
&\qquad\qquad\qquad \xrightarrow{\mathrm{id}\otimes\phi^B(\zeta)^t} \mathcal{H}_{m-n-a,n-b}\otimes\mathcal{H}^*_{m-n-a',n-b'}.
\end{aligned} \tag{6.3}$$

Here t denotes the transpose. Specifically, we define

$$\begin{aligned}
T^A(\zeta) &= \sum_{j=0}^{n}T^A_j(\zeta), \quad & T^A_j(\zeta) &= g^{(0,n)}\,\phi^A_j(\zeta)\otimes\phi^A_{n-j}(\zeta)^t,\\
T^B(\zeta) &= \sum_{j=0}^{m}T^B_j(\zeta), \quad & T^B_j(\zeta) &= g^{(0,m)}\,\phi^B_j(\zeta)\otimes\phi^B_{m-j}(\zeta)^t,
\end{aligned} \tag{6.4}$$

where the constants $g^{(0,n)}$ and $g^{(0,m)}$ are given by (4.17).

6.3 Diagonalisation of the full transfer matrices

A vacuum is, by definition, a largest eigenvalue eigenvector of the composition $\mathcal{T}(\zeta) = T^B(\zeta)\circ T^A(\zeta)$. In our alternating spin model, there are $(m-n+1)(n+1)$ degenerate vacua $|\mathrm{vac}\rangle_{a,b}\in\mathcal{H}_{a,b}\otimes\mathcal{H}^*_{a,b}$. The expressions for these vacua appear simple if we express them as elements of $\mathrm{End}(\mathcal{H})$. We conjecture that the vacua $|\mathrm{vac}\rangle_{a,b}$, and ${}_{a,b}\langle\mathrm{vac}|$ are given by

$${}_{a,b}\langle\mathrm{vac}| = |\mathrm{vac}\rangle_{a,b} = (\chi_a^{(m-n)}\chi_b^{(n)})^{-\frac{1}{2}}(-q)^D\pi_{a,b}, \tag{6.5}$$

Here, $\chi_r^{(\ell)}$ is the character

$$\chi_r^{(\ell)} = \mathrm{Tr}_{V(\lambda_r^{(\ell)})}(q^{2D}), \tag{6.6}$$

whose appearance gives the normalisation ${}_{a,b}\langle \mathrm{vac}|\mathrm{vac}\rangle_{\mathrm{a,b}} = 1$, and $\pi_{a,b} \in \mathrm{End}(\mathcal{H})$ is the projector to $\mathcal{H}_{a,b}$.

Let us consider the action of $\mathcal{T}(\zeta)$ on $|\mathrm{vac}\rangle_{a,b}$. First, note that the action of a map $\mathcal{O}_1 \otimes \mathcal{O}_2 : \mathcal{H} \otimes \mathcal{H}^* \to \mathcal{H} \otimes \mathcal{H}^*$ on an element $f \in \mathrm{End}(\mathcal{H})$ is given by $\mathcal{O}_1 \circ f \circ \mathcal{O}_2^t$. Then, using (6.4), (6.1) and properties (5.15), and (5.16) we have

$$\begin{aligned} T^A(\zeta)|\mathrm{vac}\rangle_{a,b} &= |\mathrm{vac}\rangle_{a,n-b}, & (6.7)\\ T^B(\zeta)|\mathrm{vac}\rangle_{a,b} &= |\mathrm{vac}\rangle_{m-n-a,n-b}, & (6.8) \end{aligned}$$

and hence $\mathcal{T}(\zeta)|\mathrm{vac}\rangle_{a,b} = |\mathrm{vac}\rangle_{m-n-a,b}$. To be precise about our use of the terminology 'eigenvector' or 'eigenvalue', the vacuum vector $|\mathrm{vac}\rangle_{a,b}$ is not an eigenvector of $\mathcal{T}(\zeta)$ but of $\mathcal{T}(\zeta)^2$ or $\mathcal{T}(1)^{-1}\mathcal{T}(\zeta)$. However, in the following we abuse this terminology, and call $|\mathrm{vac}\rangle_{a,b}$ a vacuum eigenvector.

Let us show how to derive (6.7). From (6.4), and (6.1) we have,

$$\begin{aligned} T^A(\zeta)|\mathrm{vac}\rangle_{a,b} &= g^{(0,n)} \sum_{j=0}^{n} (\mathrm{id} \otimes \Phi_j^{(0,n)}(\zeta))((-q)^D \otimes (-q)^D)\pi_{a,b}(\mathrm{id} \otimes \Phi_{n-j}^{(0,n)}(\zeta)), & (6.9)\\ &= g^{(0,n)} \sum_{j=0}^{n} ((-q)^D \otimes \Phi_j^{(0,n)}(\zeta)(-q)^D \Phi_{n-j}^{(0,n)}(\zeta))\pi_{a,n-b}, & (6.10)\\ &= g^{(0,n)} \sum_{j=0}^{n} ((-q)^D \otimes (-q)^D)(\mathrm{id} \otimes \Phi_j^{(0,n)}(-q^{-1}\zeta)\Phi_{n-j}^{(0,n)}(\zeta))\pi_{a,n-b}, & (6.11)\\ &= ((-q)^D \otimes (-q)^D)\pi_{a,n-b} = |\mathrm{vac}\rangle_{a,n-b}. & (6.12) \end{aligned}$$

In executing steps (6.11) and (6.12), we have used properties (5.15) and (5.16) respectively. Equation (6.8) can be shown similarly.

We will construct excited states by making use of the following operators

$$\begin{aligned} \psi_\varepsilon^{(\frac{1}{2})s}(\xi) &= \sum_\kappa \psi_{\varepsilon,\kappa}^{(\frac{1}{2})s} \xi^{-\kappa} = \Psi_\varepsilon^{*s}(\xi) \otimes \mathrm{id},\\ \psi^{(0)s,s'}(\xi) &= \sum_\kappa \psi_\kappa^{(0)s,s'} \xi^{-\kappa} = \sum_{\varepsilon=0,1} \Phi_\varepsilon^s(\xi) \otimes \Psi_\varepsilon^{*s'}(\xi), \end{aligned} \tag{6.13}$$

where Ψ_ε^{*s} and Φ_ε^s are defined by (4.9) and (4.10). Again $s, \tilde{s} = \pm$ (or equivalently ± 1).

These components act as follows.

$$\begin{aligned} \psi_{\varepsilon,\kappa}^{(\frac{1}{2})s} &: V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}) \to V(\lambda_{a+s}^{(m-n)}) \otimes V(\lambda_b^{(n)}),\\ \psi_\kappa^{(0)s,s'} &: V(\lambda_a^{(m-n)}) \otimes V(\lambda_b^{(n)}) \to V(\lambda_{a+s}^{(m-n)}) \otimes V(\lambda_{b+s'}^{(n)}). \end{aligned}$$

Using the commutation relations (5.15)–(5.19), and (2.2), we arrive at

$$\begin{aligned}
T^A(\zeta)\psi_\varepsilon^{(\frac{1}{2})\,s}(\xi)|\mathrm{vac}\rangle_{a,b} &= \psi_\varepsilon^{(\frac{1}{2})\,s}(\xi)|\mathrm{vac}\rangle_{a,n-b},\\
T^B(\zeta)\psi_\varepsilon^{(\frac{1}{2})\,s}(\xi)|\mathrm{vac}\rangle_{a,b} &= \tau(\zeta/\xi)\psi_\varepsilon^{(\frac{1}{2})\,-s}(\xi)|\mathrm{vac}\rangle_{m-n-a,n-b},\\
T^A(\zeta)\psi^{(0)\,s,\tilde{s}}(\xi)|\mathrm{vac}\rangle_{a,b} &= \tau(\zeta/\xi)\psi^{(0)\,s,-\tilde{s}}(\xi)|\mathrm{vac}\rangle_{a,n-b},\\
T^B(\zeta)\psi^{(0)\,s,\tilde{s}}(\xi)|\mathrm{vac}\rangle_{a,b} &= \psi^{(0)\,-s,-\tilde{s}}(\xi)|\mathrm{vac}\rangle_{m-n-a,n-b},
\end{aligned}$$

and hence that

$$\begin{aligned}
\mathcal{T}(\zeta)\psi_\varepsilon^{(\frac{1}{2})\,s}(\xi)|\mathrm{vac}\rangle_{a,b} &= \tau(\zeta/\xi)\psi_\varepsilon^{(\frac{1}{2})\,-s}(\xi)|\mathrm{vac}\rangle_{m-n-a,b},\\
\mathcal{T}(\zeta)\psi^{(0)\,s,\tilde{s}}(\xi)|\mathrm{vac}\rangle_{a,b} &= \tau(\zeta/\xi)\psi^{(0)\,-s,\tilde{s}}(\xi)|\mathrm{vac}\rangle_{m-n-a,b}.
\end{aligned}$$

The vectors $\psi_\varepsilon^{(\frac{1}{2})\,s}(\xi)|\mathrm{vac}\rangle_{a,b}$ and $\psi^{(0)\,s,\tilde{s}}(\xi)|\mathrm{vac}\rangle_{a,b}$ are the spin-$\frac{1}{2}$ and spin-0 eigenstates mentioned in the introduction. Note that both states are degenerate with respect to $\mathcal{T}(\zeta)$, but that $\psi_\varepsilon^{(\frac{1}{2})\,s}(\xi)|\mathrm{vac}\rangle_{a,b}$ has an eigenvalue of 1 for $T^A(\xi)$, and $\psi_\varepsilon^{(0)\,s,\tilde{s}}(\xi)|\mathrm{vac}\rangle_{a,b}$ an eigenvalue of 1 for $T^B(\xi)$. This is consistent with the Bethe Ansatz calculations for the alternating spin-$\frac{1}{2}$/spin-1 model presented in [12].

Further eigenstates may be constructed by acting with any composition of $\psi^{(\frac{1}{2})\,s_i}(\xi_i)$ and $\psi^{(0)\,s'_i,\tilde{s}'_i}(\xi'_i)$ on $|\mathrm{vac}\rangle_{a,b}$. The eigenvalues of $\mathcal{T}(\zeta)$ are the product of all the $\tau(\zeta/\xi_i)$ and $\tau(\zeta/\xi'_i)$ factors.

6.4 The S-matrix

The S-matrix for our model is specified by the exchange relations of $\psi^{(\frac{1}{2})\,s_i}(\xi_i)$ and $\psi^{(0),s'_i,\tilde{s}'_i}(\xi'_i)$ with themselves and with each other. These intertwiners are defined in terms of the intertwiners $\Phi^s(\zeta)$ and $\Psi^{*,s}(\zeta)$ of irreducible modules in (6.13). If we act with both sides on the level ℓ highest-weight module $V(\lambda_r^{(\ell)})$, then the commutation relations of the $\Phi^s(\zeta)$ and $\Psi^{*,s}(\zeta)$ are

$$\sum_{\varepsilon'_1,\varepsilon'_2}\bar{R}^{(1,1)}(\xi)_{\varepsilon_1,\varepsilon_2}^{\varepsilon'_1,\varepsilon'_2}\Phi_{\varepsilon'_1}^{s_1}(\xi_1)\Phi_{\varepsilon'_2}^{s_2}(\xi_2)=\sum_{s'_1,s'_2}\Phi_{\varepsilon_2}^{s'_2}(\xi_2)\Phi_{\varepsilon_1}^{s'_1}(\xi_1)W_\ell^I\left(\begin{array}{cc|c} r & r+s_2 & \\ r+s'_1 & r+s_1+s_2 & \xi\end{array}\right), \tag{6.14}$$

$$\Psi_{\varepsilon_1}^{*\,s_1}(\xi_1)\Psi_{\varepsilon_2}^{*\,s_2}(\xi_2)=\sum_{s'_1,s'_2,\varepsilon'_1,\varepsilon'_2}\Psi_{\varepsilon'_2}^{*\,s'_2}(\xi_2)\Psi_{\varepsilon'_1}^{*\,s'_1}(\xi_1)\bar{R}^{(1,1)}(\xi)_{\varepsilon_1,\varepsilon_2}^{\varepsilon'_1,\varepsilon'_2}W_\ell^{II}\left(\begin{array}{cc|c} r & r+s_2 & \\ r+s'_1 & r+s_1+s_2 & \xi\end{array}\right), \tag{6.15}$$

$$\Phi_{\varepsilon_1}^{s_1}(\xi_1)\Psi_{\varepsilon_2}^{*\,s_2}(\xi_2)=\sum_{s'_1,s'_2}\Psi_{\varepsilon_2}^{*\,s'_2}(\xi_2)\Phi_{\varepsilon_1}^{s'_1}(\xi_1)W_\ell^{*}\left(\begin{array}{cc|c} r & r+s_2 & \\ r+s'_1 & r+s_1+s_2 & \xi\end{array}\right). \tag{6.16}$$

Here, the sum over s'_1 and s'_2 is restricted to the values for which $s'_1+s'_2=s_1+s_2$. W_ℓ^I, W_ℓ^{II} and W_ℓ^* are given in terms of the RSOS Boltzmann weight $\overline{W}_\ell^1$ (given for example in

equation (B.2) in [5]) as follows:

$$W_\ell^I\left(\begin{array}{cc} r & s \\ u & t \end{array}\middle|\,\xi\right) = \frac{X(p^2\xi^{-2})}{X(p^2\xi^2)}\,\overline{W}_\ell^1\left(\begin{array}{cc} \lambda_r^{(\ell)} & \lambda_s^{(\ell)} \\ \lambda_u^{(\ell)} & \lambda_t^{(\ell)} \end{array}\middle|\,\xi^2\right)\xi^{\delta_{t,s+1}-\delta_{r,u-1}},$$

$$W_\ell^{II}\left(\begin{array}{cc} r & s \\ u & t \end{array}\middle|\,\xi\right) = \frac{X(\xi^{-2})}{X(\xi^2)}\,\overline{W}_\ell^1\left(\begin{array}{cc} \lambda_r^{(\ell)} & \lambda_s^{(\ell)} \\ \lambda_u^{(\ell)} & \lambda_t^{(\ell)} \end{array}\middle|\,\xi^2\right)\xi^{\delta_{t,s+1}-\delta_{r,u-1}},$$

$$W_\ell^{*}\left(\begin{array}{cc} r & s \\ u & t \end{array}\middle|\,\xi\right) = \frac{X(p\xi^{-2})}{X(p\xi^2)}\,\overline{W}_\ell^1\left(\begin{array}{cc} \lambda_r^{(\ell)} & \lambda_s^{(\ell)} \\ \lambda_u^{(\ell)} & \lambda_t^{(\ell)} \end{array}\middle|\,p^{-1}\xi^2\right)(-\xi q^{-(1+r)})^{\delta_{t,s+1}-\delta_{r,u-1}}\,q^{\delta_{r,t}},$$

where $\xi = \xi_1/\xi_2$, $X(z) = \frac{(z;p^2,q^4)_\infty(q^4z;p^2,q^4)_\infty}{(q^2z;p^2,q^4)^2_\infty}$ and $p = q^{\ell+2}$ (note that the p of [4] is equal to our p^2). Relations (6.14) and (6.15) come from [4], where they were obtained (for a homogeneous evaluation representation) by solving the q-KZ equation. We obtained (6.16) by making use of the technique mentioned in Proposition A.4 (ii) of [4] (and due originally to Okado).

Using these commutation relations, the definitions (6.13), and the unitarity property (2.2), it is then simple to show that on $V(\lambda_a^{(m-n)})\otimes V(\lambda_b^{(n)})$ (and hence on $|\mathrm{vac}\rangle_{a,b}$) we have

$$\psi_{\varepsilon_1}^{(\frac12)\,s_1}(\xi_1)\psi_{\varepsilon_2}^{(\frac12)\,s_2}(\xi_2)=\sum_{\varepsilon_1',\varepsilon_2',s_1',s_2'}\psi_{\varepsilon_2'}^{(\frac12)\,s_2'}(\xi_2)\psi_{\varepsilon_1'}^{(\frac12)\,s_1'}(\xi_1)\bar{R}^{(1,1)}(\xi)_{\varepsilon_1,\varepsilon_2}^{\varepsilon_1',\varepsilon_2'}\,W_{m-n}^{II}\left(\begin{array}{cc} a & a+s_2 \\ a+s_1' & a+s_1+s_2 \end{array}\middle|\,\xi\right), \tag{6.17}$$

$$\begin{aligned}\psi^{(0)\,s_1,\tilde{s}_1}(\xi_1)\psi^{(0)\,s_2,\tilde{s}_2}(\xi_2) = &\sum_{s_1',s_2',\tilde{s}_1',\tilde{s}_2'}\psi^{(0)\,s_2',\tilde{s}_2'}(\xi_2)\psi^{(0)\,s_1',\tilde{s}_1'}(\xi_1)\\ &\times W_{m-n}^{I}\left(\begin{array}{cc} a & a+s_2 \\ a+s_1' & a+s_1+s_2 \end{array}\middle|\,\xi\right)W_{n}^{II}\left(\begin{array}{cc} b & b+\tilde{s}_2 \\ b+\tilde{s}_1' & b+\tilde{s}_1+\tilde{s}_2 \end{array}\middle|\,\xi\right),\end{aligned} \tag{6.18}$$

$$\psi^{(0)\,s_1,\tilde{s}_1}(\xi_1)\psi_{\varepsilon_2}^{(\frac12)\,s_2}(\xi_2) = \sum_{s_1',s_2'}\psi_{\varepsilon_2}^{(\frac12)\,s_2'}(\xi_2)\psi^{(0)\,s_1',\tilde{s}_1}(\xi_1)\,W_{m-n}^{*}\left(\begin{array}{cc} a & a+s_2 \\ a+s_1' & a+s_1+s_2 \end{array}\middle|\,\xi\right). \tag{6.19}$$

Again, the sums are restricted so that $s_1'+s_2' = s_1+s_2$, and $\tilde{s}_1'+\tilde{s}_2' = \tilde{s}_1+\tilde{s}_2$.

When $n=1$, $m=2$, our model consists of alternating spin-$\frac12$ and spin-1 lines. In this case we have

$$\begin{aligned}\psi_{\varepsilon_1}^{(\frac12)}(\xi_1)\psi_{\varepsilon_2}^{(\frac12)}(\xi_2) &= -\sum_{\varepsilon_1',\varepsilon_2'}\psi_{\varepsilon_2'}^{(\frac12)}(\xi_2)\psi_{\varepsilon_1'}^{(\frac12)}(\xi_1)R^{(1,1)}(\xi)_{\varepsilon_1,\varepsilon_2}^{\varepsilon_1',\varepsilon_2'},\\ \psi^{(0)}(\xi_1)\psi^{(0)}(\xi_2) &= -\psi^{(0)}(\xi_2)\psi^{(0)}(\xi_1),\\ \psi^{(0)}(\xi_1)\psi_{\varepsilon_2}^{(\frac12)}(\xi_2) &= \tau(\xi)\psi_{\varepsilon_2}^{(\frac12)}(\xi_2)\psi^{(0)}(\xi_1).\end{aligned}$$

Here, the intertwiners act on the tensor product of level-1 irreducible highest weight modules. So, the s and $\tilde{s}$ superscripts depend solely on the choice i and j, and we suppress them. These relations are consistent with Bethe Ansatz calculations of the S-matrix for this example [12].

7 The Domain Wall Description of the Path Space and the Particle Picture

7.1 Domain walls

Let us now use $|p\rangle$ to denote a double infinite path $|p\rangle = \cdots p(2)\, p(1)\, p(0)\, p(-1)\, p(-2) \cdots$, for which

$$p(k) \in \{0, 1, \cdots, n\} \quad \text{if } k \text{ is odd,} \tag{7.1}$$

$$p(k) \in \{0, 1, \cdots, m\} \quad \text{if } k \text{ is even.} \tag{7.2}$$

Define

$$\mathcal{P} = \oplus_{a,b;a',b'} P_{a,b;a',b'}, \tag{7.3}$$

where $P_{a,b;a',b'}$ is the set

$$P_{a,b;a',b'} = \{|p\rangle;\ p(k) = \bar{p}(k; a, b), k \gg 0;\ p(k) = \bar{p}(k; a', b'), k \ll 0\}. \tag{7.4}$$

The ground state path $\bar{p}(k; a, b)$ was defined by (2.22) (note, however, that k may now be negative).

In this section, we construct a domain wall description of the space $\mathcal{P}$ and give rules for the induced crystal action on this set of domain walls.

First, we label a *domain* of a path $|p\rangle \in \mathcal{P}$ by a pair of integers (a, b), which can take the values $0 \le a \le m - n$ and $0 \le b \le n$. Suppose we start with a path $|p\rangle \in \mathcal{P}$ and try to associate a particular domain $(a(k), b(k))$ with each k, such that $p(k) = \bar{p}(k; a(k), b(k))$. There are clearly different choices of how to do this. For example, suppose we choose $k \equiv 0 \pmod 4$. Then because $\bar{p}(k; a, b) = a + b$, we could associate any of the domains $(p(k) - b, b)$, such that $0 \le p(k) - b \le m - n$ and $0 \le b \le n$, with k.

In order to fix uniquely which domain $(a(k), b(k))$ to associate with a particular k such that $p(k) = \bar{p}(k; a(k), b(k))$, we use the following rules.

(1) Choose k odd.

(2) If $n \le p(k+1) + p(k) \le m$, let

$$b(k+1) = b(k) = n - p(k), \tag{7.5}$$

$$a(k+1) = a(k) = \begin{cases} m - p(k) - p(k+1) & \text{if } k \equiv 1 \pmod 4; \\ p(k) + p(k+1) - n & \text{if } k \equiv 3 \pmod 4. \end{cases} \tag{7.6}$$

(3) If $p(k+1) + p(k) > m$, let

$$b(k+1) = p(k+1) - m + n, \quad b(k) = n - p(k), \tag{7.7}$$

$$a(k+1) = a(k) = \begin{cases} 0 & \text{if } k \equiv 1 \pmod 4; \\ m-n & \text{if } k \equiv 3 \pmod 4. \end{cases} \tag{7.8}$$

(4) If $p(k+1) + p(k) < n$, let

$$b(k+1) = p(k+1), \quad b(k) = n - p(k), \tag{7.9}$$

$$a(k+1) = a(k) = \begin{cases} m-n & \text{if } k \equiv 1 \pmod 4; \\ 0 & \text{if } k \equiv 3 \pmod 4. \end{cases} \tag{7.10}$$

By following these rules for all odd k, we can associate a unique domain $(a(k), b(k))$ for all $k \in \mathbf{Z}$. Then $p(k)$ will be given by

$$p(k) = \bar{p}(k; a(k), b(k)). \tag{7.11}$$

We can write the resulting $(a(k), b(k))_{k\in\mathbf{Z}}$ as a sequence of domains $(a_{N+1}, b_{N+1}) \cdots (a_1, b_1)$ and a sequence of integers $k_N > k_{N-1} > \cdots > k_1$. The identification is that

$$(a(k), b(k)) = (a_i, b_i) \text{ for } k_i \geq k > k_{i-1} \text{ (with } k_{N+1} = \infty, k_0 = -\infty). \tag{7.12}$$

Definition 7.1. *Let $\mathcal{D}$ be the set of elements, each of which is specified by a domain sequence $(a_{N+1}, b_{N+1}) \cdots (a_1, b_1)$ and integers $k_N > k_{N-1} > \cdots > k_1$, where $N \in \mathbf{Z}_{\geq 0}$, $0 \leq a_i \leq m-n$, $0 \leq b_i \leq n$, $(a_{i+1}, b_{i+1}) \neq (a_i, b_i)$, and*

$$k_i \in \begin{cases} 2\mathbf{Z} \cup (1 + 4\mathbf{Z}) & \textit{if} \quad a_{i+1} = a_i = 0,\ b_{i+1} > b_i, \\ 2\mathbf{Z} \cup (1 + 4\mathbf{Z}) & \textit{if} \quad a_{i+1} = a_i = m-n,\ b_{i+1} < b_i, \\ 2\mathbf{Z} \cup (3 + 4\mathbf{Z}) & \textit{if} \quad a_{i+1} = a_i = 0,\ b_{i+1} < b_i, \\ 2\mathbf{Z} \cup (3 + 4\mathbf{Z}) & \textit{if} \quad a_{i+1} = a_i = m-n,\ b_{i+1} > b_i, \\ 2\mathbf{Z} & \textit{otherwise.} \end{cases} \tag{7.13}$$

Then rules (1)-(4) and equation (7.12) specify an injection $M_1 : \mathcal{P} \to \mathcal{D}$, and (7.11) specifies a map $M_2 : \mathcal{D} \to \mathcal{P}$ which is the left inverse of M_1, i.e., $M_2 \circ M_1 = \mathrm{id}_{\mathcal{P}}$.

Proposition 7.2. *$M_1 : \mathcal{P} \to \mathcal{D}$ is a bijection.*

Proof. We will prove that the left inverse $M_2 : \mathcal{D} \to \mathcal{P}$ is an injection, from which the proposition follows. Consider two elements, $(a(k), b(k))_{k\in\mathbf{Z}}$ and $(a'(k), b'(k))_{k\in\mathbf{Z}}$ of $\mathcal{D}$ (we

can specify them in this way by making use of (7.12)). Let $k_0 \equiv 1 \pmod 4$. Then from the definition of $\mathcal{D}$, one of the following must be true

$$\begin{array}{rll} \text{I.} & a(k_0+1) = a(k_0), & b(k_0+1) = b(k_0), \\ \text{II.} & a(k_0+1) = a(k_0) = 0, & b(k_0+1) > b(k_0), \\ \text{III.} & a(k_0+1) = a(k_0) = m-n, & b(k_0+1) < b(k_0). \end{array}$$

One of three similar conditions must hold for $a'(k_0+1)$, $a'(k_0)$, $b'(k_0+1)$, and $b'(k_0)$. Under the map M_2 we have $(a(k), b(k))_{k\in\mathbf{Z}} \to \big(\bar{p}(k; a(k), b(k))\big)_{k\in\mathbf{Z}}$. The requirements that

$$\bar{p}(k_0; a(k_0), b(k_0)) = \bar{p}(k_0; a'(k_0), b'(k_0)), \tag{7.14}$$
$$\bar{p}(k_0+1; a'(k_0+1), b'(k_0+1)) = \bar{p}(k_0+1; a'(k_0+1), b'(k_0+1)) \tag{7.15}$$

are equivalent to

$$b(k_0) = b'(k_0), \tag{7.16}$$
$$a(k_0+1) - b(k_0+1) = b'(k_0+1) - a'(k_0+1) \tag{7.17}$$

respectively. Combining (7.16), (7.17), one of I, II, III for $a(k_0+1)$, $a(k_0)$, $b(k_0+1)$, $b(k_0)$ and one of the similar conditions I, II, III for $a'(k_0+1)$, $a'(k_0)$, $b'(k_0+1)$, $b'(k_0)$, we get nine possible sets of equations in eight unknowns. It is only possible to get a solution to three of these sets of equations, namely those we get when $a(k_0+1)$, $a(k_0)$, $b(k_0+1)$, $b(k_0)$ and $a'(k_0+1)$, $a'(k_0)$, $b'(k_0+1)$, $b'(k_0)$ both satisfy I, or both satisfy II, or both satisfy III. The single solution for all three sets is

$$a(k_0+1) = a'(k_0+1), \;\; a(k_0) = a'(k_0), \;\; b(k_0+1) = b'(k_0+1), \;\; b(k_0) = b'(k_0). \tag{7.18}$$

A similar argument leads to the same solution (7.18) in the case when $k_0 \equiv 3 \pmod 4$. This completes the proof. $\square$

The next step is to understand the induced crystal action on $\mathcal{D}$. If we refer to the position at which two domains meet as a *domain wall*, then the general picture is that the crystal action moves domain walls around. In order to describe this action we first identify certain types of domain wall as *elementary*. The following is a complete list of elementary walls.

$(a_{i+1},b_{i+1})(a_i,b_i)$	k_i	symbol
$(a-1,b)(a,b)$	0 mod 4	$\vert_1^-$
$(a+1,b)(a,b)$	0 mod 4	$\vert_0^+$
$(0,b+1)(0,b)$	0 mod 4	$\lfloor_0^+$
$(m-n,b-1)(m-n,b)$	0 mod 4	$\lceil_1^-$
$(a\pm 1,b\mp 1)(a,b)$	0 mod 4	$\bullet^{\pm\mp}$
$(0,b+1)(0,b)$	1 mod 4	$\lfloor_1^+$
$(m-n,b-1)(m-n,b)$	1 mod 4	$\lceil_0^-$
$(a-1,b)(a,b)$	2 mod 4	$\vert_0^-$
$(a+1,b)(a,b)$	2 mod 4	$\vert_1^+$
$(0,b-1)(0,b)$	2 mod 4	$\lfloor_1^-$
$(m-n,b+1)(m-n,b)$	2 mod 4	$\lceil_0^+$
$(a\pm 1,b\pm 1)(a,b)$	2 mod 4	$\bullet^{\pm\pm}$
$(0,b-1)(0,b)$	3 mod 4	$\lfloor_0^-$
$(m-n,b+1)(m-n,b)$	3 mod 4	$\lceil_1^+$

We write $[$ to mean either of $\lfloor$ or $\lceil$. We shall refer to $\vert_\varepsilon^s$, $[_\varepsilon^t$ as spin-$\frac{1}{2}$ elementary walls, and to $\bullet^{s,t}$ as spin-0 elementary walls.

We wish to decompose each domain wall of an element in $\mathcal{D}$ into *ordered* elementary domain walls. We use the fact that when k_i is even, $(a_{i+1},b_{i+1})(a_i,b_i)$ may be written in terms of a unique sequence of intermediate domains such that the corresponding intermediate domain walls are elementary, and ordered as

$$\vert_1\cdots\vert_1[_1\cdots[_1\bullet\cdots\bullet \quad \text{or} \quad \vert_0\cdots\vert_0[_0\cdots[_0\bullet\cdots\bullet, \tag{7.19}$$

where the $\bullet$ are taken to be of one kind only. When k_i is odd, $(a_{i+1},b_{i+1})(a_i,b_i)$ may also be written uniquely in terms of an ordered sequence of elementary domain walls of the form

$$[_1\cdots[_1 \quad \text{or} \quad [_0\cdots[_0. \tag{7.20}$$

The whole sequence of ordered elementary walls will then said to have been *normally ordered.* It is simple to prove the uniqueness of these ordered decompositions, but perhaps more illuminating to consider some examples.

1) $m=6$, $n=2$, $k_i\equiv 0 \pmod 4$:
$(0,0)(3,1)=(0,0)(1,0)(2,0)(3,0)(4,0)(3,1)\sim \vert_1^-\vert_1^-\vert_1^-\vert_1^-\bullet^{+-}$,
$(1,2)(1,0)=(1,2)(0,2)(0,1)(1,0)\sim \vert_0^+\lfloor_0^+\bullet^{-+}$,
$(4,0)(4,2)=(4,0)(4,1)(4,2)\sim \lceil_1^-\lceil_1^-$.

2) $m=6$, $n=2$, $k_i\equiv 2 \pmod 4$:
$(0,0)(3,1)=(0,0)(1,0)(2,0)(3,1)\sim \vert_0^-\vert_0^-\bullet^{--}$,

$(1,2)(1,0) = (1,2)(2,2)(3,2)(2,1)(1,0) \sim |_0^-|_0^- \bullet^{++} \bullet^{++}$,
$(4,0)(4,2) = (4,0)(3,0)(2,0)(3,1)(4,2) \sim |_1^+|_1^+ \bullet^{--} \bullet^{--}$.

3) $m=6$, $n=2$, $k_i \equiv 1 \pmod 4$:
$(0,2)(0,0) = (0,2)(0,1)(0,0) \sim \lfloor_1^+ \lfloor_1^+$,
$(4,0)(4,2) = (4,0)(4,1)(4,2) \sim \lceil_0^- \lceil_0^-$.

4) $m=6$, $n=2$, $k_i \equiv 3 \pmod 4$:
$(0,0)(0,2) = (0,0)(0,1)(0,2) \sim \lfloor_0^- \lfloor_0^-$,
$(4,2)(4,0) = (4,2)(4,1)(4,0) \sim \lceil_1^+ \lceil_1^+$.

Explicitly, the ordered walls turn out as follows.

$\underline{(a_2,b_2)(a_1,b_1) \text{ at } k \equiv 0 \pmod 4.}$

$$(|_0^+)^{a_2}(\lfloor_0^+)^{b_2-b_1-a_1}(\bullet^{-+})^{a_1} \quad \text{if } b_2-b_1>a_1; \tag{7.21}$$

$$(|_0^+)^{a_2+b_2-a_1-b_1}\begin{cases}(\bullet^{-+})^{b_2-b_1} & (b_2 \geq b_1);\\ (\bullet^{+-})^{b_1-b_2} & (b_2 \leq b_1)\end{cases} \quad \text{if } a_2+b_2 \geq a_1+b_1 \geq b_2; \tag{7.22}$$

$$(|_1^-)^{a_1+b_1-a_2-b_2}\begin{cases}(\bullet^{-+})^{b_2-b_1} & (b_2 \geq b_1);\\ (\bullet^{+-})^{b_1-b_2} & (b_2 \leq b_1)\end{cases} \quad \text{if } m-n+b_2 \geq a_1+b_1 \geq a_2+b_2; \tag{7.23}$$

$$(|_1^-)^{m-n-a_2}(\lceil_1^-)^{a_1+b_1-b_2-m+n}(\bullet^{+-})^{m-n-a_1} \quad \text{if } b_1-b_2>m-n-a_1. \tag{7.24}$$

$\underline{(0,b_2)(0,b_1) \text{ at } k \equiv 1 \pmod 4.}$

$$(\lfloor_1^+)^{b_2-b_1} \tag{7.25}$$

$\underline{(m-n,b_2)(m-n,b_1) \text{ at } k \equiv 1 \pmod 4.}$

$$(\lceil_0^-)^{b_1-b_2} \tag{7.26}$$

$\underline{(a_2,b_2)(a_1,b_1) \text{ at } k \equiv 2 \pmod 4.}$

$$(|_1^+)^{a_2}(\lfloor_1^-)^{b_1-b_2-a_1}(\bullet^{--})^{a_1} \quad \text{if } b_1-b_2>a_1; \tag{7.27}$$

$$(|_1^+)^{b_1-a_1-b_2+a_2}\begin{cases}(\bullet^{--})^{b_1-b_2} & (b_2 \leq b_1);\\ (\bullet^{++})^{b_2-b_1} & (b_2 \geq b_1)\end{cases} \quad \text{if } b_2 \geq b_1-a_1 \geq b_2-a_2; \tag{7.28}$$

$$(|_0^-)^{b_2-a_2-b_1+a_1}\begin{cases}(\bullet^{--})^{b_1-b_2} & (b_2 \leq b_1);\\ (\bullet^{++})^{b_2-b_1} & (b_2 \geq b_1)\end{cases} \quad \text{if } b_2-a_2 \geq b_1-a_1 \geq b_2-m+n; \tag{7.29}$$

$$(|_0^-)^{m-n-a_2}(\lceil_0^+)^{b_2-b_1+a_1-m+n}(\bullet^{++})^{m-n-a_1} \quad \text{if } b_2-b_1>m-n-a_1. \tag{7.30}$$

$\underline{(0,b_2)(0,b_1) \text{ at } k \equiv 3 \pmod 4.}$

$$(\lfloor_0^-)^{b_1-b_2} \tag{7.31}$$

$\underline{(m-n,b_2)(m-n,b_1) \text{ at } k\equiv 3 \pmod 4.}$

$$(\lceil_1^+)^{b_2-b_1} \tag{7.32}$$

After normally ordering all the walls in an element $|d\rangle \in \mathcal{D}$, the rules for the crystal action are relatively simple (we give the rule for the action of $\tilde{f}_i$, the action of $\tilde{e}_i$ can be reconstructed in terms of the inverse of this). Suppose we have a total of N elementary spin-$\frac{1}{2}$ walls with subscript $\varepsilon_N, \varepsilon_{N-1}, \cdots, \varepsilon_1$ (and any number of spin-0 walls). Now consider the vector $[\varepsilon_N]^{(1)} \otimes [\varepsilon_{N-1}]^{(1)} \otimes \cdots \otimes [\varepsilon_1]^{(1)} \in (B^{(1)})^{\otimes N}$. The operator $\tilde{f}_i$ acts on $[\varepsilon_N]^{(1)} \otimes [\varepsilon_{N-1}]^{(1)} \otimes \cdots \otimes [\varepsilon_1]^{(1)}$ by changing a single $\varepsilon_j \to 1-\varepsilon_j$ (or by sending the vector to zero). Which ε_j is changed depends on whether $i=0$ or 1. The action of $\tilde{f}_i$ on the element $|d\rangle$ is to change only the single elementary spin-$\frac{1}{2}$ wall with the corresponding ε_j index (or it sends the path to zero). The change that occurs for this elementary domain wall depends on its type and position k in the following way:

$$\begin{array}{cccccccc} & k+2 & k+1 & k & & k+2 & k+1 & k \end{array}$$

$$\begin{array}{cccccccc} (\cdots) & \underbrace{\bullet\cdots\bullet}_{c} & & |_{\varepsilon}(\cdots) & \to & (\cdots)|_{1-\varepsilon}\underbrace{\bullet\cdots\bullet}_{c} & & (\cdots) \end{array} \tag{7.33}$$

$$\begin{array}{cccccccc} (\cdots) & \underbrace{\bullet\cdots\bullet}_{c} & & |_{\varepsilon}(\cdots) & \to & (\cdots)[_{1-\varepsilon}\underbrace{\bullet\cdots\bullet}_{c-1} & & (\cdots) \end{array} \tag{7.34}$$

$$\begin{array}{cccccccc} & & (\cdots) & [_{\varepsilon}(\cdots) & \to & & (\cdots)[_{1-\varepsilon} & (\cdots) \end{array} \tag{7.35}$$

$$\begin{array}{cccccccc} (\cdots) & \underbrace{\bullet\cdots\bullet}_{c} & [_{\varepsilon}(\cdots) & & \to & (\cdots)|_{1-\varepsilon}\underbrace{\bullet\cdots\bullet}_{c+1} & (\cdots) & \end{array} \tag{7.36}$$

$$\begin{array}{cccccccc} (\cdots) & \underbrace{\bullet\cdots\bullet}_{c} & [_{\varepsilon}(\cdots) & & \to & (\cdots)[_{1-\varepsilon}\underbrace{\bullet\cdots\bullet}_{c} & (\cdots) & \end{array} \tag{7.37}$$

Here, we have taken k to be even. Also, for (7.34) and (7.37), we are assuming that the domain appearing on the left of $\bullet\cdots\bullet$ is at the appropriate boundary, i.e., $(0,*)$ if $[_{1-\varepsilon}$ that may appear at the position $k+2$ is a $\lfloor$ and $(m-n,*)$ if it is a $\lceil$.

Before showing how these rules for the crystal action were obtained, let us give some simple examples of how this general rule works. The following two examples capture the two possible ways in which a spin-$\frac{1}{2}$ wall can pass a spin-0 wall under the crystal action.

First, suppose $m=6$, $n=2$ and that we have an element of $\mathcal{D}$ described by 3 domains $(4,2)(3,1)(4,1)$ and positions $k_2=2$, $k_1=0$. The $(4,2)(3,1)$ wall at $k_2=2$ is a $\bullet^{++}$ elementary wall. The $(3,1)(4,1)$ wall at $k_1=0$ is a $|_1^-$ elementary wall. Using M_2 (as specified by (7.11)), we can write out the section of path in which these walls lie. The path is

$$\cdots \quad 6 \quad 0 \quad 2 \quad 0 \quad 6 \quad 0 \ \bullet^{++} \ 2 \quad 1 \ |_1^- \ 5 \quad 1 \quad \cdots \tag{7.38}$$

Using the above rules for the crystal action on elementary domain walls we find that $\tilde{f}_1$ sends this path to 0, and $\tilde{f}_0$ sends it to

$$\cdots \quad 6 \quad 0 \quad 2 \quad 0 \quad 6 \quad 0\ \lceil^+_0\ 1 \quad 1 \quad 5 \quad 1 \quad \cdots \tag{7.39}$$

Here, we have used (7.34). This is a path associated with a single domain wall $(4,2)(4,1)$ at $k_1 = 2$. Working out the subsequent action of $\tilde{f}_1$, $\tilde{f}_0$, and so on, one finds,

$$\begin{array}{cccccccccccccc}
 & \cdots & 6 & 0 & 2 & 0 & 6 & 0\ \bullet^{++} & 2 & 1\ |^-_1 & 5 & 1 & \cdots \\
\xrightarrow{\tilde{f}_0} & \cdots & 6 & 0 & 2 & 0 & 6 & 0\ \lceil^+_0 & 1 & 1 & 5 & 1 & \cdots \\
\xrightarrow{\tilde{f}_1} & \cdots & 6 & 0 & 2 & 0 & 6\ \lceil^+_1 & 1 & 1 & 1 & 5 & 1 & \cdots \\
\xrightarrow{\tilde{f}_0} & \cdots & 6 & 0 & 2 & 0|^+_0\bullet^{-+}5 & & 1 & 1 & 1 & 5 & 1 & \cdots \\
\xrightarrow{\tilde{f}_1} & \cdots & 6 & 0\ |^+_1 & 3 & 0\ \bullet^{-+} & 5 & 1 & 1 & 1 & 5 & 1 & \cdots
\end{array} \tag{7.40}$$

etc. Here we have just shown the elementary wall decomposition at each stage. The final sequence of domains is $(4,2)(3,2)(4,1)$. Notice, the action of $\tilde{f}_0$ on the fourth line used (7.36) and not (7.37). This is because the domain on the left of the (non-existent) $\bullet\cdots\bullet$ is at the boundary, but not the relevant one.

Now consider the rather similar example when $m = 6, n = 2$ and we have an element $|d\rangle \in \mathcal{D}$ described by 3 domains $(3,2)(2,1)(3,1)$ and positions $k_2 = 2$, $k_1 = 0$. Again, the $(3,2)(2,1)$ wall at $k_2 = 2$ is a $\bullet^{++}$ elementary wall and the $(2,1)(3,1)$ wall at $k_1 = 0$ is a $|^-_1$ elementary wall. Using the rules for the arrows, we get the following sequence.

$$\begin{array}{cccccccccccccc}
 & \cdots & 5 & 0 & 3 & 0 & 5 & 0\ \bullet^{++} & 3 & 1\ |^-_1 & 4 & 1 & \cdots \\
\xrightarrow{\tilde{f}_0} & \cdots & 5 & 0 & 3 & 0 & 5 & 0|^-_0\bullet^{++}2 & & 1 & 4 & 1 & \cdots \\
\xrightarrow{\tilde{f}_1} & \cdots & 5 & 0 & 3 & 0\ |^-_1 & 6 & 0\ \bullet^{++} & 2 & 1 & 4 & 1 & \cdots \\
\xrightarrow{\tilde{f}_0} & \cdots & 5 & 0\ |^-_0 & 3 & 0 & 6 & 0\ \bullet^{++} & 2 & 1 & 4 & 1 & \cdots
\end{array} \tag{7.41}$$

The final sequence of domains is $(3,2)(4,2)(3,1)$. We see that the spin-0 wall remains fixed in this case, whereas in (7.40) it was moved 2 spaces to the left by the passage of the spin-$\frac{1}{2}$ wall.

Let us now show how the rules (7.33)–(7.37) were obtained. We shall consider $\tilde{f}_1$ only. Let $|p\rangle$ be any path. Following the rule (3.3), we associate a sequence of 1's and 0's to each $p(k)$. Then, using the usual rule, we simplify it in such a way that each domain wall carries $(1)^c$ or $(0)^c$. This is determined locally at each wall and called the localisation. It is convenient to think that there always exists a domain wall between $k+1$:odd and k:even. If it is not a real one, the localisation is trivial, i.e., $c = 0$. Let us explain this more carefully, starting with two examples.

$k \equiv 1 \pmod 4$ and $n \leq p(k+1) + p(k) \leq m$.

Recalling the process for fixing the domains, given at the beginning of this section, we

see that, in this case, there is no wall between $p(k+1)$ and $p(k)$. The domain is (a,b), where $p(k+1) = m-n-a+b$ and $p(k) = n-b$ (see (7.5) and (7.6)). We have

$$(1)^{m-n-a+b}(0)^{n+a-b}(1)^{n-b}(0)^{b} \sim (1)^{m-n-a+b}(0)^{a+b}. \tag{7.42}$$

Distribute $(1)^{m-n-a+b}$ to the left wall, and $(0)^{a+b}$ to the right.

<u>$k \equiv 1 \pmod 4$ and $p(k+1)+p(k) < n$.</u>
The domain changes at the centre. The domains are given by $(m-n, b_2)(m-n, b_1)$ with $p(k+1) = b_2$ and $p(k) = n - b_1$. We have $b_2 < b_1$.

$$(1)^{b_2}(0)^{m-b_2}(1)^{n-b_1}(0)^{b_1} \sim (1)^{b_2}(0)^{m-n+2b_1-b_2}. \tag{7.43}$$

Distribute $(1)^{b_2}$ to the left, $(0)^{b_1-b_2}$ to the centre, and $(0)^{m-n+b_1}$ to the right.

We carry out a similar procedure for all other cases. Now, consider a wall between $k+1$:odd and k:even. Suppose $(0)^{c_1}$ is distributed from the left and $(1)^{c_2}$ from the right. If $c_1 > c_2$, the localisation is $(0)^{c_1-c_2}$. If $c_1 = c_2$, there is no (real) wall. If $c_1 < c_2$, the localisation is $(1)^{c_2-c_1}$. For a wall between $k+1$:even and k:odd, the localisation is already given in the form $(0)^c$ or $(1)^c$. In fact, we have the following simple rule (for the $\tilde{f}_1$ case).

$$\begin{array}{ccc}
\text{domain} & \text{position} & \text{localisation} \\
(a_2,b_2)(a_1,b_1) & k \equiv 0 & (0)^{a_2+b_2}(1)^{a_1+b_1} \\
(0,b_2)(0,b_1) & k \equiv 1 & (1)^{b_2-b_1} \\
(m-n,b_2)(m-n,b_1) & k \equiv 1 & (0)^{b_1-b_2} \\
(a_2,b_2)(a_1,b_1) & k \equiv 2 & (0)^{m-n-a_2+b_2}(1)^{m-n-a_1+b_1} \\
(0,b_2)(0,b_1) & k \equiv 3 & (0)^{b_1-b_2} \\
(m-n,b_2)(m-n,b_1) & k \equiv 3 & (1)^{b_2-b_1}
\end{array} \tag{7.44}$$

We now consider the action of $\tilde{f}_1$. Suppose it acts on the part of a path $x = p(k+1)$ and $y = p(k)$ with $k \equiv 3 \pmod 4$. Suppose that by the action of $\tilde{f}_1$ we have the change:

$$x \to x+1. \tag{7.45}$$

In the 1 and 0 notation, this part is equivalent to starting from

$$(1)^{x}(0)^{m-x}(1)^{y}(0)^{n-y} \tag{7.46}$$

and changing the leftmost 0 in $(0)^{m-x}$ to 1. It implies $m-x > y$. We have two cases.

<u>$n \leq x+y < m$.</u>
Both x and y belong to the same domain, say, (a_2, b_2). We have $x = a_2 + b_2$ and $y = n - b_2$. Therefore, we have

$$a_2 + n = x + y < m. \tag{7.47}$$

Let (a_1, b_1) be the domain on the right of y, (a_3, b_3) on the left of x.

It is necessary that the localisation at the wall between (a_2, b_2) and (a_1, b_1) is $(0)^{c_1}$ with $c_1 > 0$. Therefore, we have $a_1 - b_1 - a_2 + b_2 > 0$ and this wall is of the form (7.29) or (7.30). Because of (7.47), we see that the number of $|_0$ is at least one. It is also necessary that the localisation at the wall between (a_3, b_3) and (a_2, b_2) is $(1)^{c_2}$ with $c_2 \geq 0$. Therefore, we have $a_3 + b_3 - a_2 - b_2 \geq 0$ and this wall is of the form (7.23) or (7.24).

The change (7.45) is equivalent to the change $a_2 \to a_2 + 1$. Using the explicit wall descriptions (7.29), (7.30), (7.23), and (7.24), we see that it corresponds to (7.33) or (7.34).

$\underline{x + y < n.}$
There is a wall between x and y. We have the domains $(a_3, b_3)(0, b_2)(0, b_1)$, where x and y belong to $(0, b_2)$ and $(0, b_1)$, respectively. We have $x = b_2$ and $y = n - b_1$, and therefore $b_1 - b_2 = n - x - y > 0$.

The wall between $(0, b_2)$ and $(0, b_1)$ is of the form (7.31). It is also necessary that the localisation at the wall between (a_3, b_3) and $(0, b_2)$ is of the form $(1)^{b_2 - a_3 - b_3}$. Therefore, we have $b_2 - a_3 - b_3 \geq 0$, and, in particular, $b_2 - b_3 \geq 0$. This wall is of the form (7.23) (the lower line) or (7.24).

The change (7.45) is equivalent to the change $b_2 \to b_2 + 1$. It corresponds to (7.36) (for (7.23)) or (7.37) (for (7.24)).

We may also consider the $\tilde{f}_1$ action as sending $y \to y+1$. This will bring about the remaining case $x+y \geq m$ and corresponds to (7.35). The case $k \equiv 1 \pmod 4$ may be similarly analysed to confirm the results (7.33)–(7.37).

Before ending this section, let us consider one consequence of the rules for the crystal action on $\mathcal{D}$. Let $|d\rangle \in \mathcal{D}$ have normally ordered elementary domain walls at positions $k_K, \cdots, k_1$. Define

$$n(k_i, |_\varepsilon) = n(k_i, \bullet) = -k_i/2, \quad n(k_i, [_\varepsilon) = -k_i. \tag{7.48}$$

It is simple to check from the rules for the crystal action that $\sum_{i=1}^{K} n(k_i, t_i)$ decreases by 1 under the action of $\tilde{f}_i$, and increases by 1 under the action of $\tilde{e}_i$. So, the action of the principal grading operator ρ is given by

$$\rho(|d\rangle) = \sum_{i=1}^{K} n(k_i, t_i)|d\rangle, \tag{7.49}$$

where t_i refers to the 'type' $|_\varepsilon$, $[_\varepsilon$, or $\bullet$ of the elementary wall.

7.2 The particle picture

In describing a path $|p\rangle \in \mathcal{P}$ in terms of either local spin variables $p(k)$ or a sequence of domains and domain walls, we have been using the *local picture*. We shall now go on to explain the *particle picture* of the space $\mathcal{P} \simeq \mathcal{D}$. Let $\mathcal{D}_{a,b;a',b'}$ denote the range of the restricted map $M_1|_{\mathcal{P}_{a,b;a',b'}} : \mathcal{P}_{a,b;a',b'} \to \mathcal{D}$. As a crystal, $\mathcal{D}_{a,b;a',b'}$ will decompose into a (usually infinite) number of connected components. We wish to understand these connected components as the crystals created by the creation operators $\psi^{(\frac{1}{2})s}_{\varepsilon,\kappa}$ and $\psi^{(0)s,t}_{\kappa}$ of spin-$\frac{1}{2}$ and spin-0 particles. We call this the *particle picture*. The operators $\psi^{(\frac{1}{2})s}_{\varepsilon,\kappa}$ and $\psi^{(0)s,t}_{\kappa}$ will be given as the $q \to 0$ limit of the corresponding operators defined in Section 6 (we conjecture that this limit is well-defined). In the particle picture, any sequence of the operators $\psi^{(\frac{1}{2})s}_{\kappa}$ and $\psi^{(0)s,t}_{\kappa}$ is allowed, but with the condition that the corresponding sequence of the highest weights (represented by (a_i, b_i)) satisfies $0 \le a_i \le m-n$ and $0 \le b_i \le n$. This condition will always be assumed when we talk of a sequence of these operators. However, they are not linearly independent because of the commutation relations (6.17)–(6.19) of the particle creation operators. The particle pictures for the pure spin-$\frac{1}{2}$ model, pure spin-$\frac{n}{2}$ models and RSOS fusion models were constructed in references [1], [29] and [30] respectively.

Before looking at the space spanned by the particles, we prepare some details about affine crystals. Suppose we have a $U_q(\widehat{sl}_2)$ crystal B which takes weights in $P_{cl} = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1$. Then the affinization of this crystal, denoted by $\mathrm{Aff}(B)$, takes weights in $P = \mathbf{Z}\Lambda_1 \oplus \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\delta$ (See [14] for a definition). Here, we use $\mathrm{Aff}(B)$ defined in the principal gradation. For example, $\mathrm{Aff}(B^{(1)})$ is given by either of the following diagrams.

$$\begin{tikzcd}[column sep=small]
\vdots & \\
{[0]\otimes[2\kappa+1]} \arrow[dr, "\tilde f_1"] & \\
& {[1]\otimes[2\kappa]} \arrow[dl, "\tilde f_0"'] \\
{[0]\otimes[2\kappa-1]} \arrow[dr, "\tilde f_1"] & \\
& {[1]\otimes[2\kappa-2]} \\
\vdots &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\vdots & \\
& {[1]\otimes[2\kappa+1]} \arrow[dl, "\tilde f_0"'] \\
{[0]\otimes[2\kappa]} \arrow[dr, "\tilde f_1"] & \\
& {[1]\otimes[2\kappa-1]} \arrow[dl, "\tilde f_0"'] \\
{[0]\otimes[2\kappa-2]} & \\
\vdots &
\end{tikzcd}$$

Let us now consider the states spanned by just one particle. From the definition (6.13) and the remarks following (4.10), we see that $\psi^{(\frac{1}{2})s}_{\varepsilon,\kappa}$ is meaningful only if $s \cdot (-1)^{\varepsilon} = (-1)^{\kappa}$ and that $\psi^{(0)s,t}_{\kappa}$ is meaningful only if $-s \cdot t = (-1)^{\kappa}$. Considering the degree given by (7.49) also, we identify

$$\begin{aligned} |^{s}_{\varepsilon} \text{ at } k \quad &\longleftrightarrow \quad \psi^{(\frac{1}{2})s}_{\varepsilon,-\frac{k}{2}}, \\ \bullet^{s,t} \text{ at } k \quad &\longleftrightarrow \quad \psi^{(0)s,t}_{-\frac{k}{2}}. \end{aligned} \tag{7.50}$$

Recalling the rules for the crystal action on the elementary walls, we see that each set of $\psi^{(\frac{1}{2})s}_{\varepsilon,\kappa}$

with s fixed and other indices satisfying $s \cdot (-1)^\varepsilon = (-1)^\kappa$ brings about a crystal isomorphic to $\mathrm{Aff}(B^{(1)})$. Each set of $\psi_\kappa^{(0)s,t}$ with both s and t fixed is a crystal isomorphic to $\mathrm{Aff}(B^{(0)})$.

To consider spaces spanned by more than one particle, we have to study the linear dependence relations in the particle picture more carefully. We take the $q \to 0$ limit of the relations (6.17)–(6.19) and write out the results componentwise. When acting on (a, b), we have

$$\begin{aligned}
\psi_{\varepsilon,\kappa}^{(\frac{1}{2})s}\psi_{\varepsilon',\kappa'}^{(\frac{1}{2})s'} &= -\psi_{\varepsilon,\kappa'+\nu}^{(\frac{1}{2})s}\psi_{\varepsilon',\kappa-\nu}^{(\frac{1}{2})s'} \quad \text{with } \nu = \delta_{s,s'} + \delta_{\varepsilon,\varepsilon'}, && (7.51)\\
\psi_\kappa^{(0)s,t}\psi_{\kappa'}^{(0)s',t'} &= -\psi_{\kappa'+\nu}^{(0)s,t}\psi_{\kappa-\nu}^{(0)s',t'} \quad \text{with } \nu = \delta_{s,s'} + \delta_{t,t'}, && (7.52)\\
\psi_\kappa^{(0)s,t}\psi_{\varepsilon,\kappa'}^{(\frac{1}{2})s} &= \psi_{\varepsilon,\kappa'}^{(\frac{1}{2})s}\psi_\kappa^{(0)s,t}, && (7.53)\\
\psi_\kappa^{(0)+,t}\psi_{\varepsilon,\kappa'}^{(\frac{1}{2})-} &= \begin{cases} \psi_{\varepsilon,\kappa'}^{(\frac{1}{2})-}\psi_\kappa^{(0)+,t} & \text{if } a \neq m-n, \\ \psi_{\varepsilon,\kappa'+1}^{(\frac{1}{2})+}\psi_{\kappa-1}^{(0)-,t} & \text{if } a = m-n, \end{cases} && (7.54)\\
\psi_\kappa^{(0)-,t}\psi_{\varepsilon,\kappa'}^{(\frac{1}{2})+} &= \begin{cases} \psi_{\varepsilon,\kappa'}^{(\frac{1}{2})+}\psi_\kappa^{(0)-,t} & \text{if } a \neq 0, \\ \psi_{\varepsilon,\kappa'+1}^{(\frac{1}{2})-}\psi_{\kappa-1}^{(0)+,t} & \text{if } a = 0. \end{cases} && (7.55)
\end{aligned}$$

Relations (7.53)–(7.55) tell us that we may always order the particles so that all the $\psi^{(0)}$ are to the right of all the $\psi^{(\frac{1}{2})}$. Relation (7.51) shows that $\psi_{\varepsilon,\kappa}^{(\frac{1}{2})s}\psi_{\varepsilon',\kappa-\nu}^{(\frac{1}{2})s'} = 0$. A little more scrutiny at (7.51) shows that we may always order any nonzero $\psi_{\varepsilon,\kappa}^{(\frac{1}{2})s}\psi_{\varepsilon',\kappa'}^{(\frac{1}{2})s'}$ so that $\kappa < \kappa'$, or in the case $(s, \varepsilon) = (s', \varepsilon')$, $\kappa \leq \kappa'$. A similar statement is true for $\psi_\kappa^{(0)s,t}\psi_{\kappa'}^{(0)s',t'}$. We have shown:

Proposition 7.3. *Any sequence of M spin-$\frac{1}{2}$ particles $\psi_{\varepsilon,\kappa}^{(\frac{1}{2})s}$ and N spin-0 particles $\psi_\kappa^{(0)s,t}$ initiating and terminating at two given domains may be written in the form*

$$\psi_{\varepsilon_1,\kappa_1'}^{(\frac{1}{2})s_1'} \cdots \psi_{\varepsilon_M,\kappa_M'}^{(\frac{1}{2})s_M'}\psi_{\kappa_1}^{(0)s_1,t_1} \cdots \psi_{\kappa_N}^{(0)s_N,t_N}, \tag{7.56}$$

modulo sign, if it is not equal to zero. Here, we require the indices to satisfy

$$\kappa_i' < \kappa_{i+1}' \ \textit{or}\ (\kappa_i', s_i', \varepsilon_i) = (\kappa_{i+1}', s_{i+1}', \varepsilon_{i+1}), \tag{7.57}$$
$$\kappa_i < \kappa_{i+1} \ \textit{or}\ (\kappa_i, s_i, t_i) = (\kappa_{i+1}, s_{i+1}, t_{i+1}). \tag{7.58}$$

A sequence of particles of the form given by this proposition will be called *separately ordered*. The name comes from the way the spin-$\frac{1}{2}$ particles and spin-0 particles have been grouped separately. This is to be contrasted with the *normally ordered* sequence to be defined in Section 7.3.

Let us now consider the vector space which is spanned by the sequence of particles. We fix M, the number of spin-$\frac{1}{2}$ operators $\psi_{\varepsilon,\kappa}^{(\frac{1}{2})s}$, N, the number of spin-0 operators $\psi_\kappa^{(0)s,t}$, and the initial and final domains. We will denote the space by $\mathcal{A}$. We do not impose the commutation relations (7.51)–(7.55) in $\mathcal{A}$. As before, we identify $\psi_{\varepsilon,\kappa}^{(\frac{1}{2})s}$ and $\psi_\kappa^{(0)s,t}$ with the

elements of $\mathrm{Aff}(B^{(1)})$ and $\mathrm{Aff}(B^{(0)})$. Hence the monomial basis of $\mathcal{A}$ is a crystal isomorphic to a union of mixed tensor products of M-many $\mathrm{Aff}(B^{(1)})$ and N-many $\mathrm{Aff}(B^{(0)})$. We call it the *crystal part of* $\mathcal{A}$.

The subspace spanned by the relations will be denoted by $\mathcal{R}$. It is easy to prove, using the tensor product rule for crystals bases, that the set of relations (7.51)–(7.55) is preserved under the crystal action. Hence, the monomial basis of $\mathcal{A}/\mathcal{R}$ is given a crystal structure. We call it the *crystal part of* $\mathcal{A}/\mathcal{R}$. We are interested in this crystal structure.

Denote by $\mathcal{S}$, the set of separately ordered sequence of particles. It is easy to show that $\mathcal{S}$ is also preserved under the crystal action. Hence, $\mathcal{S}$ is a subcrystal of the crystal part of $\mathcal{A}$. We aim to show that $\mathcal{S}$ forms a basis of $\mathcal{A}/\mathcal{R}$ so that the crystal $\mathcal{S}$ is, in fact, the crystal part of $\mathcal{A}/\mathcal{R}$.

We first give a partial ordering to the set of particles. Two particles are said to satisfy $\psi^A < \psi^B$ if and only if $\psi^A \neq \psi^B$ and $\psi^A\psi^B$ is separately ordered. Then the monomial basis elements of $\mathcal{A}$ are given the lexicographical order using the order on the particles. We define an action of S_{M+N}, the symmetric group of order $M+N$, on $\mathcal{A}$. Since all the relations (7.51)–(7.55) are of the form $\psi^A\psi^B = \pm\psi^C\psi^D$, we may define the action of the transposition $\sigma_i = (i, i+1)$ on a sequence of particles by substituting $\psi^A\psi^B$ at the i-th and $(i+1)$-th position with the appropriate $\pm\psi^C\psi^D$. It is easy to show that this defines an action of S_{M+N} on $\mathcal{A}$. We prove two lemmas concerning these definitions.

Lemma 7.4. *Suppose $M+N \geq 2$. Let $A = \psi^{A_1}\cdots\psi^{A_{M+N}}$ and $B = \psi^{B_1}\cdots\psi^{B_{M+N}}$. If A is separately ordered, and $\sigma_1\sigma_2\cdots\sigma_{r-1}A = \pm B$ $(r \leq M+N)$, then $\psi^{A_1} < \psi^{B_1}$.*

Proof. It suffices to show this for the case $r = M+N$. We use induction on r. For $r = 2$, this may be done by checking each case. So suppose $r > 2$. Let $\sigma_{r-1}A = \pm\psi^{A_1}\cdots\psi^{A_{r-2}}\psi^C\psi^{B_r}$. We know from the $r = 2$ case that $\psi^{A_{r-1}} < \psi^C$. Hence, $\psi^{A_1}\cdots\psi^{A_{r-2}}\psi^C$ is separately ordered. We may now apply induction hypothesis to conclude $\psi^{A_1} < \psi^{B_1}$. □

Lemma 7.5. *Suppose $M+N \geq 2$. Let $A = \psi^{A_1}\cdots\psi^{A_{M+N}}$ and $B = \psi^{B_1}\cdots\psi^{B_{M+N}}$. If A is separately ordered, $\pi \in S_{M+N}$ is different from the identity element, and $\pi A = \pm B$, then $A < B$.*

Proof. We use induction on $M+N$. This is easy to check when $M+N = 2$. If $M+N > 2$ and $\pi(1) = 1$, then we may apply the induction hypothesis to $A' = \psi^{A_2}\cdots\psi^{A_{M+N}}$ and $B' = \psi^{B_2}\cdots\psi^{B_{M+N}}$. So suppose $\pi(r) = 1$ with $r > 1$. Then, we may write $\pi = \pi'\sigma_1\sigma_2\cdots\sigma_{r-1}$ for some $\pi' \in S_{M+N}$ with $\pi'(1) = 1$. But, then Lemma 7.4 shows, $\psi^{A_1} < \psi^{B_1}$ and hence $A < B$. □

The next easy corollary to this lemma shows that the expression (7.56) is unique for each product of particles different from zero.

Corollary 7.6. *Let x be separately ordered and choose any $\pi \in S_{M+N}$. Then, $\pi(x)$ is separately ordered if and only if $\pi = \mathrm{id}$.*

We can now finally prove:

Proposition 7.7. *The set of separately ordered elements, $\mathcal{S}$, forms a basis for $\mathcal{A}/\mathcal{R}$.*

Proof. By Proposition 7.3, it suffices to show the linear independence of $\mathcal{S}$. Let $(\ |\ )$ denote the natural orthonormal bilinear form on $\mathcal{A}$. Define $\bar{x} = \sum_{\pi \in S_{M+N}} \pi(x)$ for any $x \in \mathcal{A}$. Noting

$$\mathcal{R} = \mathrm{Span}\{\pi(x) - x; \pi \in S_{M+N}, x \in \mathcal{A}\},$$

we have $(\bar{x}|\mathcal{R}) = 0$ for any $x \in \mathcal{S}$. Hence, $(\bar{x}|\cdot)$ defines a linear functional on $\mathcal{A}/\mathcal{R}$. Using Corollary 7.6, we may easily check that $(\bar{x}|y)_{x,y\in\mathcal{S}} = \delta_{x,y}$. This proves that the set $\mathcal{S}$ is linearly independent. $\square$

So the space described by the particles initiating and terminating at given domains is the crystal $\mathcal{S}$ of separately ordered sequence of particles. We have obtained a clear view of the particle picture given in terms of the affine crystals $\mathrm{Aff}(B^{(1)})$ and $\mathrm{Aff}(B^{(0)})$.

7.3 Connection between the local and particle pictures

Let us first describe a map from the domain wall description to the particle picture. We have already identified the walls $|_\varepsilon^s$ and $\bullet^{s,t}$ with the particles in (7.50). Writing out the domain wall description in the path form, at $k \equiv 0 \pmod 4$, we can check

$$(0, b+1) \lfloor_0^+ (0, b) = (0, b+1) \bullet^{-+} |_0^+ (0, b).$$

We may similarly write other $[$ at even k as a combination of $\bullet^{s,t}$ and $|_\varepsilon^t$. With this and the identification (7.50), we map

$$\begin{aligned} \lfloor_\varepsilon^t \text{ at even } k &\quad\longmapsto\quad \psi^{(0)-,t}_{-\frac{k}{2}} \psi^{(\frac{1}{2})+}_{\varepsilon,-\frac{k}{2}}, \\ \lceil_\varepsilon^t \text{ at even } k &\quad\longmapsto\quad \psi^{(0)+,t}_{-\frac{k}{2}} \psi^{(\frac{1}{2})-}_{\varepsilon,-\frac{k}{2}}. \end{aligned} \tag{7.59}$$

To map the remaining four elementary walls, we return to example (7.40).

$$\begin{array}{ccccccccccccccccccc}
 & \cdots & 0 & & 2 & & 0 & & 6 & & 0 & \bullet^{++} & 2 & & 1 & |_1^- & 5 & \cdots & \psi^{(0)+,+}_{-1} \psi^{(\frac{1}{2})-}_{1,0} \\
\xrightarrow{\tilde{f}_0} & \cdots & 0 & & 2 & & 0 & & 6 & & 0 & \lceil_0^+ & 1 & & 1 & & 5 & \cdots & \psi^{(0)+,+}_{-1} \psi^{(\frac{1}{2})-}_{0,-1} \\
\xrightarrow{\tilde{f}_1} & \cdots & 0 & & 2 & & 0 & & 6 & \lceil_1^+ & 1 & & 1 & & 1 & & 5 & \cdots & \boxed{\ ?\ } \\
\xrightarrow{\tilde{f}_0} & \cdots & 0 & & 2 & & 0 & |_0^+ \bullet^{-+} & 5 & & 1 & & 1 & & 1 & & 5 & \cdots & \psi^{(\frac{1}{2})+}_{0,-2} \psi^{(0)-,+}_{-2} \\
\xrightarrow{\tilde{f}_1} & \cdots & 0 & |_1^+ & 3 & & 0 & \bullet^{-+} & 5 & & 1 & & 1 & & 1 & & 5 & \cdots & \psi^{(\frac{1}{2})+}_{1,-3} \psi^{(0)-,+}_{-2}
\end{array} \tag{7.60}$$

This time, we have written the particles to the right using (7.50) and (7.59). What should go in the box? Coming down from the top, we can guess it to be $\psi^{(0)+,+}_{-1}\psi^{(\frac{1}{2})-}_{1,-2}$. Going up from the bottom, it should be $\psi^{(\frac{1}{2})+}_{1,-1}\psi^{(0)-,+}_{-2}$. We are dealing with the $a = m - n$ case, and as (7.54) with $t = +$ shows, they are actually equal. Generalising this, we map

$$\begin{aligned} \lfloor^t_\varepsilon \text{ at odd } k &\quad\longmapsto\quad \psi^{(\frac{1}{2})+}_{\varepsilon,-\frac{1}{2}(k-1)}\psi^{(0)-,t}_{-\frac{1}{2}(k+1)}, \\ \lceil^t_\varepsilon \text{ at odd } k &\quad\longmapsto\quad \psi^{(\frac{1}{2})+}_{\varepsilon,-\frac{1}{2}(k-1)}\psi^{(0)-,t}_{-\frac{1}{2}(k+1)}. \end{aligned} \tag{7.61}$$

We have defined a map from the domain wall description to the particle picture.

We now define the inverse map. To do this, we construct a new basis of $\mathcal{A}/\mathcal{R}$. We say a sequence of particles is *normally ordered* if each successive pair is one of the following:

(1) $\psi^{(0)s_1,t_1}_{\kappa_1}\psi^{(0)s_2,t_2}_{\kappa_2}$ where $\kappa_1 < \kappa_2$ or $(\kappa_1, s_1, t_1) = (\kappa_2, s_2, t_2)$.

(2) $\psi^{(1/2)s_1}_{\varepsilon_1,\kappa_1}\psi^{(1/2)s_2}_{\varepsilon_2,\kappa_2}$ where $\kappa_1 < \kappa_2$ or $(\kappa_1, \varepsilon_1, s_1) = (\kappa_2, \varepsilon_2, s_2)$.

(3) $\psi^{(1/2)s'}_{\varepsilon,\kappa'}\psi^{(0)s,t}_{\kappa}$ where $\kappa' \le \kappa$.

(4) $\psi^{(1/2)-s}_{\varepsilon,\kappa}\psi^{(0)s,t}_{\kappa-1}$ where these are placed at the boundary, i.e., for $s = +$, it acts on the domain $(0, *)$, for $s = -$, it acts on the domain $(m - n, *)$.

(5) $\psi^{(0)s,t}_{\kappa}\psi^{(1/2)s'}_{\varepsilon,\kappa'}$ where $\kappa < \kappa'$.

(6) $\psi^{(0)-s,t}_{\kappa}\psi^{(1/2)s}_{\varepsilon,\kappa}$ where these are placed at the boundary, i.e., for $s = +$, it acts on the domain $(0, *)$, for $s = -$, it acts on the domain $(m - n, *)$.

The set of normally ordered sequence of particles will be denoted by $\mathcal{N}$. The relations (7.51)–(7.55) show that we may always bring any sequence of particles to a normally ordered sequence. The linear independence of the normally ordered sequence may be proved as in the proof for Proposition 7.3. So the normally ordered sequences form a basis for $\mathcal{A}/\mathcal{R}$. The set of normally ordered sequences of particles, $\mathcal{N}$, is certainly a crystal, the crystal action being "first, act as an element of $\mathcal{A}$, then, normally order." The map from the particle picture to the local picture may now be taken by first applying the inverse of (7.59) and (7.61) to (6) and (4), respectively, and then applying (7.50) to the remaining particles. It is easy to check that the image is an ordered sequence of elementary domain walls. The defined map is certainly inverse to the map from the local picture to the particle picture defined earlier.

Theorem 7.8. *The local picture and the particle picture are isomorphic as crystals.*

Proof. It suffices to show that the two maps defined in this section respect the crystal structures. So, let us study the rules for the crystal action on $\mathcal{N}$. We shall consider $\tilde{f}_1$ only. The action of $\tilde{f}_1$ will change some $\psi^{(\frac{1}{2})s}_{0,\kappa'}$ to $\psi^{(\frac{1}{2})s}_{1,\kappa'-1}$. After this change the product may not

be normally ordered. In that case, we must normally order it by using the commutation relations. The rules come out as follows:

If the product contains $\psi_{\kappa}^{(0)-s,+}\psi_{0,\kappa}^{(\frac{1}{2})s}$ at the boundary, i.e., it acts on the domain $(0,*)$ for the $s=+$ case and $(m-n,*)$ for the $s=-$ case, the change is

$$\psi_{\kappa}^{(0)-s,+}\psi_{0,\kappa}^{(\frac{1}{2})s} \to \psi_{1,\kappa}^{(\frac{1}{2})-s}\psi_{\kappa-1}^{(0)s,+}. \tag{7.62}$$

Otherwise and if the product contains $(\psi_{\kappa-1}^{(0)-s,-})^c\psi_{0,\kappa}^{(\frac{1}{2})s}$ for some $c\geq 1$ and the domain on the left of this part of the product is at the boundary, the change is

$$(\psi_{\kappa-1}^{(0)-s,-})^c\psi_{0,\kappa}^{(\frac{1}{2})s} \to \psi_{\kappa-1}^{(0)-s,-}\psi_{1,\kappa-1}^{(\frac{1}{2})s}(\psi_{\kappa-1}^{(0)-s,-})^{c-1}. \tag{7.63}$$

Otherwise, let $c\geq 0$ be the maximal integer such that $(\psi_{\kappa-1}^{(0)s',s's})^c\psi_{0,\kappa}^{(\frac{1}{2})s}$ is contained in the product. Then, the change is

$$(\psi_{\kappa-1}^{(0)s',s's})^c\psi_{0,\kappa}^{(\frac{1}{2})s} \to \psi_{1,\kappa-1}^{(\frac{1}{2})s}(\psi_{\kappa-1}^{(0),s',s's})^c. \tag{7.64}$$

In the domain wall language, the case (7.62) corresponds to (7.35). The case (7.63) corresponds to (7.34) and (7.37). The last case (7.64) corresponds to (7.33) and (7.36). □

We have thus related the path space $\mathcal{P}$ with a crystal given explicitly in terms of $\mathrm{Aff}(B^{(0)})$ and $\mathrm{Aff}(B^{(1)})$. Namely, we have established the crystal isomorphisms between $\mathcal{P}$ and $\mathcal{D}$, $\mathcal{D}$ and $\mathcal{N}$, $\mathcal{N}$ and $\mathcal{S}$. And the crystal $\mathcal{S}$ is given as a union of subcrystals of $\mathrm{Aff}(B^{(1)})^{\otimes M}\otimes\mathrm{Aff}(B^{(0)})^{\otimes N}$.

8 Summary

Let us summarise very briefly the main results of our analysis of infinite-volume alternating-spin vertex models. We identify the space on which the transfer matrices of the alternating spin-$\frac{m}{2}$/ spin-$\frac{n}{2}$ model act as the direct sum of

$$\begin{aligned}&\mathrm{Hom}_{\mathbf{C}}\Big(V(\lambda_a^{(m-n)})\otimes V(\lambda_b^{(n)}), V(\lambda_{a'}^{(m-n)})\otimes V(\lambda_{b'}^{(n)})\Big)\\ &\simeq V(\lambda_{a'}^{(m-n)})\otimes V(\lambda_{b'}^{(n)})\otimes\Big(V(\lambda_a^{(m-n)})\otimes V(\lambda_b^{(n)})\Big)^*.\end{aligned} \tag{8.1}$$

The transfer matrices themselves are constructed in terms of certain $U_q'(\widehat{sl}_2)$ intertwiners defined on this space (see (6.4)). These transfer matrices are diagonalised by making use of another set of intertwiners given by (6.13). The vacua are given by $(-q)^D$; the excited states are multi-particle states consisting of a number of spin-0 particles and a number of spin-$\frac{1}{2}$ particles. The two-particle S-matrices are given by (6.17) to (6.19).

In [13], we show how to construct correlation functions of these models. We derive there the relation between simple correlation functions of the alternating model and those of the

pure spin-$\frac{n}{2}$ and pure spin-$\frac{m}{2}$ models. In this, and in the diagonalisation of the transfer matrix, we make use of the commutativity of one of our intertwiners (see Section 5 of the current paper) with the action of the deformed Virasoro algebra considered in [15].

In Sections 3 and 7 we consider the crystal limit (i.e., $q \to 0$ limit) of our model in detail. In this limit, the corner transfer matrix acts diagonally on the (half-infinite) path space $P_{a,b}$ associated with a particular boundary condition (a, b). We prove that there is a crystal isomorphism $P_{a,b} \simeq B(\lambda_a^{(m-n)}) \otimes B(\lambda_b^{(n)})$. We go on to consider the double infinite path space $\mathcal{P}$. We construct a crystal isomorphism between this space and the space $\mathcal{D}$ defined in terms of domain walls. $\mathcal{P}$ and $\mathcal{D}$ are both considered as *local picture* descriptions of the space. We then construct two *particle picture* descriptions, $\mathcal{N}$ and $\mathcal{S}$, by making use of the $q \to 0$ limit of the intertwiners which diagonalise our transfer matrix. We finally establish an equivalence between $\mathcal{P}$ in the local picture and $\mathcal{S}$ in the particle picture. The latter, in turn, has a description in terms of tensor products of the crystals $\mathrm{Aff}(B^{(0)})$ and $\mathrm{Aff}(B^{(1)})$.

The observations in this paper and in [13] might be applied and extended in various directions. Two of them are:

(1) It is possible to derive difference equations for correlation functions and form factors of the alternating spin model using techniques analogous to those described in [3]. It should also be possible to evaluate these quantities by making use of the free field realisation of $U_q(\widehat{sl}_2)$.

(2) The approach should generalise in a straightforward manner to alternating spin models with three or more different alternating spins.

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